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Introduction to Option Pricing in a Securities Market – III: Gaussian Approximation

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In this part of the lecture notes on securities trading we aim at the limiting transition from a binary market of part I towards the Black-Scholes model of a Gaussian market in which the stock price develops as the geometric Brownian motion.

1. Introduction

1.1. Outline

In this paper we consider the situation in which a binary securities market allows for the Gaussian approximation to be carried out in section 4.5. For the necessary material on binary markets (see section 4 below) we refer to the previous papers [18] and [17], section 3, hereafter called part I and part II, respectively. Both have appeared in the previous issues of *CWI Quarterly* devoted to *Mathematics of Finance*. The first of these issues contains also the paper [41] which we would recommend for a quick review on rudiments of stochastic calculus.

Throughout the present paper the basic conditions B and S are assumed, see the next section. They restrict the limiting behaviour of the asset prices. Condition B on the bond is the same as in part II, condition 3.1.1, while condition S on the stock differs completely from condition 3.1.2 in part II. It is in fact the adaptation of the conditions well-known in probability theory, under which a random walk admits a diffusion approximation (see e.g. [8], [21] and various papers in [43]). But since the reader is not supposed to be familiar with advanced methods of the probability theory, the presentation is kept at the same low technical level as in the previous parts by means of certain unsophisticated algebraic considerations. The emphasis then will be on phenomenological understanding the cash flow mechanism in the market under consideration, that is shown to reveal strong similarity to physical Brownian motion (more generally, to diffusion with drift) or to the molecular mechanism of heat flows, cf sections 2 and 3. For more details on the economical laws governing the market, we refer to the seminal paper [3] by Black and Scholes. The results obtained in this manner in the sections 4.5 and 5, are of heuristic nature (cf [9], [25] and [48]), for the full rigour would require higher technical level of the general theory of stochastic processes, cf [11], [12], [16], [19], [20], and [46] – [48]. For further reading we would recommend a selective list of papers [1], [2], [7], [27], [28], [30], [31], [38] and [39] (some of these papers deviate considerably from the present context and carry the reader far afield). Due to growing interest in both research and teaching a number of introductory papers and books are appeared in different languages, e.g. [5], [10], [13], [32], [36] and [40].

The point of view taken in the present paper is somewhat different: an attempt is made to facilitate reading without an advanced probabilistic prerequisite. For instance, section 2 introduces the reader to Brownian motion and its original mathematical model by Wiener. The relation to diffusion and thermal conductance is discussed in section 3. Usage of heat equations for the description of the cash flow dynamics in section 5 is preceded by proposition 4.3 that asserts an approximate heat equation for value processes in binary markets (somewhat in the spirit of Kac's paper on random walk in [43]). Moving towards the continuous-time model, in section 4.5 the approximation is discussed to the option pricing formula for the European call option. The final result is the celebrated Black-Scholes formula (5.20), cf [3], [24] or [29], section 5.8. In section 5 the Black-Scholes model is described in which the stock price process is assumed to be a geometrical Brownian motion. The theory is developed along the same lines as in the previous parts: the properties of the possible states of the discounted stock prices asserted in proposition 5.1 are shared by every discounted value process for a self-financing strategy (see proposition 5.2) and all these stem from the fact that the Gaussian transition probability density u in (5.5) and (5.11) satisfies the heat equation (3.2). In part II we have a similar situation: the Poisson distribution possesses the property asserted in lemma 4.3.1 and therefore we have (4.1.14) and (4.2.12). The counterpart in part I of these are (3.4.4) and (3.6.1), respectively. These are basically the arguments for the completeness of our markets; cf part I, proposition 4.3.3, part II, proposition 4.3.4, and the present part, proposition 5.4. In these propositions we construct the hedging strategies against any wealth desired at the terminal date T. Finally, pricing of contingent claims is accomplished by the procedure described at the end of section 4.

1.2. Basic conditions

As in part I and part II, section 3, it is assumed that in a securities market two assets, called the bond and stock, are traded during the time interval $[0, T]$. New prices on both assets are announced at certain fixed trading times, say $t_0 < t_1 < \cdots < t_N$ where $t_0 = 0$ is the current date and $t_N = T$ the terminal date. The prices on the bond and stock announced at the n^{th} trading time t_n with $n \in \{0, 1, \ldots, N\}$ are denoted by B_n^N and S_n^N, respectively. For simplicity, we restrict our attention to the special case of markets where new prices are announced regularly so that trading times are equidistant, given by $t_n = \frac{nT}{N}, n = 0, 1, \ldots, N$, and the lengths of the trading periods $\Delta t_n = t_n - t_{n-1}$ for $n \in \{1, \ldots, N\}$ are all given by $\Delta t_n = \frac{T}{N}$ (in fact one can proceed without this specification, however at the expense of some details which we want to avoid). The formulations are indeed somewhat simplified. For instance, the corresponding price processes $B^N = \{B_t^N\}_{t \in [0,T]}$ and $S^N = \{S_t^N\}_{t \in [0,T]}$ are defined in the entire time interval $[0, T]$ by $B^N(t) = B^N_{\lfloor tN/T \rfloor}$ and $S^N(t) = S^N_{\lfloor tN/T \rfloor}$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Put $B^N(0) = 1$ and $S^N(0) = s$ for simplicity, where s is a certain positive number. The discounted stock price process is denoted as in the previous parts by $\grave{S}^N = \{\grave{S}_t^N\}_{t \in [0,T]}$ with $\grave{S}^N(t) = \frac{S^N(t)}{B^N(t)}$.

The bond is a riskless asset and the price process B^N evolves along a prescribed piecewise constant trajectory, while the stock is a risky asset and the price process S^N is allowed to evolve along 2^N different trajectories of the same type. These trajectories are specified by the binary transition scheme of part I, section 3.1. They all start from the same fixed state s, the current state of the stock price $s = s_{10}^N > 0$. Further, the whole price tree is uniquely determined by two offsprings at each trading time. If at t_{n-1} with $n \in \{1, \ldots, N\}$ the stock price was in state $s^N_{k,n-1}$, then at the consecutive trading time t_n it is announced either in state $s^N_{2k,n}$ or $s^N_{2k-1,n}$ with $s^N_{2k,n} > s^N_{2k-1,n} > 0$. Hence if $t \in [t_n, t_{n+1})$ with some $n \in \{0, 1, \ldots, N\}$ the stock price $S^N(t)$ may occupy one of the states $\{s_k^N(t)\}_{k=1,\ldots,2^n}$ with $s_k^N(t) \equiv s_{kn}^N$. During the first period $[t_0, t_1)$, for instance, the stock price stays in the current state $s > 0$. At the terminal date $t_N = T$ the stock price $S^N(T)$ may occupy one of 2^N states $\{s_k^N(T)\}_{k=1,\ldots,2^N}$. In this case we also say that the stock price evolves along the k^{th} trajectory. In order to describe the stock price development along this particular trajectory, we specify the stock price state at each $t \in [t_n, t_{n+1})$ for $n = 1, \ldots, N$ by the identity $s^N_{k_n}(t) \equiv s^N_{k_n,n}$ where $k_n = \lceil \frac{k-1}{2^{N-n}} \rceil$, cf part I, definition (2.1.4), or part II, formula (3.1.5). We have used the notation $\lceil x \rceil$ for the smallest integer exceeding x.

It is supposed throughout the present paper that the number N of the trading times is very large, and the possibilities are sought for approximating the option pricing formulas of part I. To this end, we let $N \to \infty$. We can expect in the limit sensible results if only the grid $\{t_0, t_1, \ldots, t_N\}$ of trading times becomes finer and finer in the sense that the mesh width of the grid tends to zero as $N \to \infty$ (the mesh width is the maximal length of the trading periods). But this is certainly the case, as we have already chosen for the equidistant grid with the constant length $\frac{T}{N}$ of the trading periods. Further, the asset prices ought to be made dependent on the index N in a certain special manner. See the conditions B and S formulated below in terms of corresponding cumulative return processes $\mathcal{R}^N = \{\mathcal{R}_t^N\}_{t\in[0,T]}$ and $R^N = \{R_t^N\}_{t\in[0,T]}$ with

$$\mathcal{R}^N(t) = \int_0^t \frac{dB_u^N}{B_{u-}^N} = \sum_{u\in(0,t]} \frac{\Delta B_u^N}{B_{u-}^N} \quad \text{and} \quad R^N(t) = \int_0^t \frac{dS_u^N}{S_{u-}^N} = \sum_{u\in(0,t]} \frac{\Delta S_u^N}{S_{u-}^N}.$$

For more details see part II, section 3.1. Concerning the bond we assume precisely the same asymptotic behaviour as in part II.

Condition B. As $N \to \infty$ the increase of the return process on the bond over each trading period becomes proportional to the length of this period: for each $n = 1, \dots, N$

$$\frac{\mathcal{R}^N(t_n) - \mathcal{R}^N(t_{n-1})}{t_n - t_{n-1}} = r + \varrho_n^N$$

where $r > 0$ is a positive constant, while ϱ_n^N is a *negligible remainder term.*

See part II, remark 3.1, for the exact meaning of the later term. Obviously, condition B means that at the trading time t_n with $n \in \{1, \dots, N\}$ the return on the bond

$$\rho_n^N \equiv \Delta\mathcal{R}^N(t_n) = \mathcal{R}^N(t_n) - \mathcal{R}^N(t_{n-1}) \tag{1.1}$$

is asymptotically proportional to the length of the preceding period:

$$\rho_n^N = (r + \varrho_n^N)\Delta t_n \sim r\Delta t_n. \tag{1.2}$$

The sign $\sim$ indicates that the ratio of the two sides tends to unity. As is shown in part II, section 3, we have for each $t \in [0, T]$ that

$$\mathcal{R}^N(t) \sim rt \quad \text{and} \quad B^N(t) \sim e^{rt}, \tag{1.3}$$

i.e. asymptotically, the cumulative return process on the bond is assumed to increase with a constant interest rate r.

The conditions on the behaviour of the returns on stock $\Delta R^N(t_n) = R^N(t_n) - R^N(t_{n-1})$ at trading times $\{t_n\}_{n=1,\dots,N}$ are formulated as in part II in terms of their states

$$r_{kn}^N \equiv \frac{s_{kn}^N}{s_{k_{n-1},n-1}^N} - 1, \quad k = 1, \dots, 2^n, \tag{1.4}$$

where $k_{n-1} = \lceil \frac{k}{2} \rceil$ like in part II, formula (3.1.14). But the present conditions are completely different.

Condition S. At the trading time t_n with some $n \in \{1, \dots, N\}$ the return on the stock $\Delta R^N(t_n)$ is in one of the 2^n states

$$r_{kn}^N = \begin{cases} \sigma\sqrt{\Delta t_n} + (a + \eta_{kn}^N)\Delta t_n & \text{if } k \text{ is even} \\ -\sigma\sqrt{\Delta t_n} - (b + \eta_{kn}^N)\Delta t_n & \text{if } k \text{ is odd} \end{cases} \tag{1.5}$$

where $\sigma > 0$, a and b are some constants, while $\{\eta_{kn}^N\}_{k=1,\ldots,2^n}$ are negligible remainder terms as $N \to \infty$.

The negligibility of these remainder terms is understood as in part II, remark 3.1.3. Using the same sign $\sim$ as above we may express (1.5) in the following form

$$r_{kn}^N \sim \begin{cases} \sigma\sqrt{\Delta t_n} + a\Delta t_n & \text{if } k \text{ is even} \\ -\sigma\sqrt{\Delta t_n} - b\Delta t_n & \text{if } k \text{ is odd.} \end{cases} \tag{1.6}$$

If condition B holds as well, then at the trading time t_n the states $\{\grave{r}_{kn}^N\}_{k=1,\ldots,2^n}$ of the discounted return

$$\Delta \grave{R}^N(t) = \frac{\Delta \grave{S}_t^N}{\grave{S}_{t-}^N} = \frac{\Delta R^N(t) - \Delta \mathcal{R}^N(t)}{1 + \Delta \mathcal{R}^N(t)} \tag{1.7}$$

(cf part II, formula (3.1.13)) are approximated as follows. Due to (1.7) it follows from (1.2) and (1.6) that

$$\grave{r}_{kn}^N \sim \begin{cases} \sigma\sqrt{\Delta t_n} + (a - r)\Delta t_n & \text{if } k \text{ is even} \\ -\sigma\sqrt{\Delta t_n} - (b + r)\Delta t_n & \text{if } k \text{ is odd.} \end{cases} \tag{1.8}$$

By obvious reasons, the parameter $\sigma > 0$ determining the amplitude of the leading terms in these formulas is often called *volatility* (as well as *diffusion coefficient* or *thermal diffusivity*, depending on the context).

To grasp the idea behind the approximations in sections 4.5 and 5, observe the following. Like in the previous parts, let the even state indices correspond to the upward displacements, and the odd indices to the downward displacements. If now the same weight $\frac{1}{2}$ are assigned to both of these displacements, then in virtue of (1.8) the average return is approximated by

$$\frac{1}{2}(\grave{r}_{2k,n}^N + \grave{r}_{2k-1,n}^N) \sim \mu\Delta t_n \tag{1.9}$$

with a constant

$$\mu = \frac{1}{2}(a - b) - r \tag{1.10}$$

called the *drift coefficient*, since the drift μt in (5.3) and thereafter is in fact the accumulation of instant drifts on the right hand side of (1.9). Let us now average differently to get rid of the drift. Namely, let us correct the weights for upward and downward displacements to be $\frac{1}{2}(1 - \frac{\mu}{\sigma}\sqrt{\Delta t_n})$ and $\frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta t_n})$, respectively. This results in the zero mean on the right hand side: the average return is approximated by

$$\frac{1}{2}(1 - \frac{\mu}{\sigma}\sqrt{\Delta t_n})\grave{r}_{2k,n}^N + \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta t_n})\grave{r}_{2k-1,n}^N \sim \mu\Delta t_n - \mu\Delta t_n = 0. \tag{1.11}$$

In section 4.2 the weights just used occur again in formula (4.9), to serve as the approximation to so-called *risk neutral probabilities.* The usage of the last term is in clear connection with the fact expressed by (1.11) that the drift is eliminated by suitable averaging.

Next, we look at the terms proportional to $\sqrt{\Delta t_n}$ that are all the same in (1.5), (1.6) and (1.8). The sequence of the corresponding upward and downward displacements (as the index n runs trough $\{0, 1, \ldots, N\}$) are symmetrical and form the so-called *symmetrical random walk.* This is a mathematical model of the hypothetical situation in which a minute particle, immersed in a liquid, suffers many collisions with the molecules of the medium. These molecules, being in thermal motion, impart energy and momentum to the particle, so that it undergoes very irregular and erratic motion. If we imagine the collisions regularly spread in time with intervals equal $\Delta t_n = \frac{T}{N}$ that results in either upward or downward displacements of equal chance with small steps of size $\sigma\sqrt{\Delta t_n}$, then we end up in the situation under consideration. Since the number of molecules N is very large, it is convenient to let $N \to \infty$ and to look for suitable approximations. One of the central results of probability theory tells us that since the displacements are of considerably larger order of magnitude $\sim \sqrt{\Delta} t_n$ then the length Δt_n of the time interval, degeneracies are excluded and the limiting process turns out to be mathematical Brownian motion; see the previous section for relevant references. This process, denoted in the present paper by $\mathcal{W}$, will occur in the definitions (5.2) and (5.3), presenting the risky component in securities market models with continuous-time trading. Section 2 is devoted to Wiener's original construction of Brownian motion (or Wiener process, as it is often called) and to various properties of its trajectories, although it may be not so easy to imagine their appearance. But if we try to imagine a very long realization of our symmetrical random walk plotted on a graph with regular small time intervals and with displacements per time interval proportional to the square root of its length, then we are led to expect that the trajectory of the limiting process, although continuous, has an infinite number of small spikes in any finite interval and is therefore non-differentiable. This is indeed the case, see section 2.4 for further comments.

1.3. Gaussian distribution

In the present paper an important rôle is played by the so-called *probability integral*

$$G(x) = \int_{-\infty}^{x} g(y)dy \tag{1.12}$$

with the density

$$\frac{dG(x)}{dx} \equiv g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{1.13}$$

Due to the property

$$\int_{-\infty}^{\infty} g(y)dy = 1 \tag{1.14}$$

this is a probability distribution, called the standard *Gaussian* or *normal* distribution. Generally, the Gaussian distribution $G(\cdot;\mu,\sigma^2)$ with two parameters $\mu \in (-\infty,\infty)$ and $\sigma^2 \in (0,\infty)$ called the *expectation* and *variance* respectively, is defined by $G(x;\mu,\sigma^2) = G(\frac{x-\mu}{\sigma})$, with the density

$$\frac{dG(x;\mu,\sigma^2)}{dx} \equiv g(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

This density will frequently occur in our considerations through the function $u(\cdot,\cdot)$ of space coordinate $x \in (-\infty,\infty)$ and time $t \in (0,\infty)$, defined by

$$u(x,t) \equiv g(x, x_0 + \mu t, \sigma^2 t), \tag{1.15}$$

where μ and σ^2 are certain parameters, while x_0 is a certain initial site so that

$$\lim_{t\downarrow 0} u(x,t) = u(x,0) = \delta(x - x_0). \tag{1.16}$$

Here δ is Dirac's delta with the following reproducing property: for any bounded continuous function f

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)dx = f(x_0). \tag{1.17}$$

It is indeed not hard to see that for a such f

$$\lim_{t\downarrow 0} \int_{-\infty}^{\infty} u(x,t) f(x)dx = f(x_0).$$

To this end let $t \downarrow 0$ in the integral

$$\frac{1}{\sqrt{2\pi t}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0-\mu t)^2}{2t\sigma^2}} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} f(x_0 + \mu t + y\sigma\sqrt{t})dy,$$

where the substitution $\frac{x-x_0-\mu t}{\sigma\sqrt{t}} = y$ is made. Since the limit can be taken under the integration sign, from (1.13) and (1.14) we get the desired $f(x_0)$.

For brevity, we will use the following notations for partial derivatives:

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x} \quad \text{and} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}. \tag{1.18}$$

2. Brownian motion

2.1. Wiener's construction

In this section we will discuss Wiener's measure-theoretical analysis of physical Brownian motion that had been discovered at about a century earlier by Brown in 1820's, who made microscopic observations on the minute particles contained in the pollen of plants suspended in a liquid. He observed the highly irregular motion of these particles and made the first attempt to interpret this strange phenomenon. However, the true cause of the motion became known much later. It was understood that highly irregular and erratic displacements arise from thermal motion of the molecules of the liquid in which the particles are immersed, as the result of an extremely large number of collusions with these molecules. One of the first attempts to establish the mathematical framework for Brownian motion was undertaken at around the year 1900 by Bachelier, whose goal in his thesis on "theory of speculation" was to develop methods for option valuation (see [24], p 217, for a short description of these ideas). Few years later Einstein proposed the mathematical theory which we will touch upon in section 3.1. Meanwhile, in this section 2 we focus on Wiener's ideas as presented in his later books [44] and [45]. First, we will describe a set of trajectories along which the mathematical Brownian motion is allowed to evolve in time (called below Brownian paths, for brevity). Then we indicate various properties of this set of functions with which one is able to establish a sophisticated integration theory. We will start with setting up a mapping between certain sets of functions called quasi-intervals, and certain subintervals of the unit interval $0 \leq \alpha \leq 1$. This will be done in such a manner that the obtained functions $w(\cdot,\cdot)$, measurable in both arguments (t,α), will be continuous in time t for almost every α. These will be trajectories of our Brownian motion.

2.2. Quasi-intervals

Let $\mathcal{C}_0$ be a class of all real valued functions of time which start from the origin, i.e. if $f \in \mathcal{C}_0$, then $f(0) = 0$. We are now going to define particular subsets of $\mathcal{C}_0$, called *quasi-intervals* for a reason to become clear soon. Let n be any positive integer, and let $\mathbf{t}_n = \{t_j\}_{j=1,\ldots,n}$ be a set of n instants so that $0 < t_1 < \ldots < t_n \leq T$. These are n points on the t-axis. Through each of these points we pass the line perpendicular to the t-axis. On each such line, say j^{th} one, $j = 1,\ldots,n$, we choose an interval ι_j of the real axis. A quasi-interval $I_n(\mathbf{t}_n;\iota_1,\ldots,\iota_n)$ consists of all real-valued functions $f \in \mathcal{C}_0$ whose values at t_j are confined to ι_j, i.e. $f(t_j) \in \iota_j$, $j = 1,\ldots,n$. For example, take $n = 2$ and $\mathbf{t}_2 = \{\frac{1}{2}T, T\}$ and consider the following four quasi-intervals:

$$I_2(\mathbf{t}_2;\iota_{\ell_1},\iota_{\ell_2}), \quad \ell_1,\ell_2 = 1,2, \tag{2.1}$$

with

$$\iota_1 = [-\infty,0], \quad \iota_2 = (0,\infty]. \tag{2.2}$$

Obviously, these four quasi-intervals partitioning the entire class $\mathcal{C}_0$, since each particular pair is disjoint (no function can belong to two different quasi-intervals) and at the same time their union coincides with $\mathcal{C}_0$ (each function necessarily belongs to one of these quasi-intervals). For instance, $I_2(\mathbf{t}_2;\iota_1,\iota_1)$ consists of functions from $\mathcal{C}_0$ whose values at $t = \frac{1}{2}T$ and $t = T$ are non-positive, i.e. if $f \in I_2(\mathbf{t}_2;\iota_1,\iota_1)$, then $f(\frac{1}{2}T) \leq 0$ and $f(T) \leq 0$. In this section we will be only interested in such sets of quasi-intervals, partitioning $\mathcal{C}_0$. Moreover, starting from the above partition, we will construct a sequence of finer and finer partitions of a special type. At the first stage just described the subindex n was equal 2. At the next stage it will be equal 2^2, then 2^3 and so forth. The number of quasi-intervals involved at each stage will increase rapidly: starting from 4, at the n^{th} stage it will become $(2^n)^{2^n}$. But let us first describe the second stage. Take $n = 4$ and $\mathbf{t}_4 = \{\frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T, T\}$. This $\mathbf{t}_4$ contains the previous $\mathbf{t}_2$, of course. Put

$$\iota_1 = \left[-\infty, \tan\left(-\frac{\pi}{4}\right)\right], \quad \iota_2 = \left(\tan\left(-\frac{\pi}{4}\right), 0\right],$$
$$\iota_3 = \left(0, \tan\frac{\pi}{4}\right], \quad \iota_4 = \left(\tan\frac{\pi}{4}, \infty\right].$$

This refines the previous segmentation (2.2) of a real line (drawn perpendicularly to the t-axis). Recall that $\tan x$ is a monotonically increasing function of $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with $\tan 0 = 0$ and $\tan\left(\pm\frac{\pi}{2}\right) = \pm\infty$. Note that this concrete choice is immaterial, since any other monotonically increasing real-valued function will do as well. Define now a partition of $\mathcal{C}_0$ by the 4^4 quasi-intervals

$$I_4(\mathbf{t}_4; \iota_{\ell_1}, \ldots, \iota_{\ell_4}), \quad \ell_1, \ldots, \ell_4 = 1, \ldots, 4. \tag{2.3}$$

Clearly, this is a partition finer then the previous one since, for instance,

$$\bigcup_{\ell_1, \ell_3 = 1, \ldots, 4} \ \bigcup_{\ell_2, \ell_4 = 1, 2} I_4\left(\mathbf{t}_4; \iota_{\ell_1}, \ldots, \iota_{\ell_4}\right) = I_2(\mathbf{t}_2; \iota_1, \iota_1),$$

and each quasi-interval of the previous stage is obtained by the union of a certain number of quasi-intervals of the next stage.

Carrying on in this manner, at the n^{th} stage we fix the set of 2^n dyadic instants $\mathbf{t}_{2^n} = \left\{\frac{j}{2^n}T\right\}_{j=1,\ldots,2^n}$ and partition $\mathcal{C}_0$ by the following quasi-intervals:

$$I_{2^n}\left(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}}\right), \quad \ell_1, \ldots, \ell_{2^n} = 1, \ldots, 2^n,$$

where

$$\iota_1 = \left[-\infty, \tan\frac{(1 - 2^{n-1})\pi}{2^n}\right],$$

and

$$\iota_\ell = \left(\tan\frac{(\ell - 1 - 2^{n-1})\pi}{2^n}, \tan\frac{(\ell - 2^{n-1})\pi}{2^n}\right], \ \ell = 2, 3, \ldots, 2^n.$$

For example, $I_{2^n}\left(\mathbf{t}_{2^n}; \iota_1, \ldots, \iota_1\right)$ consists of all real-valued functions $f \in \mathcal{C}_0$ whose values at the 2^n dyadic instants $t = \frac{1}{2^n}T, \frac{2}{2^n}T, \ldots, T$ are restricted by the following inequalities:

$$f\left(\frac{j}{2}T\right) \leq \tan\frac{\left(1 - 2^{n-1}\right)\pi}{2^n}, \quad j = 1, \ldots, 2^n.$$

Of course, we have in this section purposely chosen for the exponential increase of dyadic instants of successive partitioning, in order to guarantee the convergence of certain procedures which will be set up below.

2.3. Equicontinuity

For each $n = 1, 2, \ldots$ we have constructed $(2^n)^{2^n}$ quasi-interval partitioning the entire class $\mathcal{C}_0$, since they do not intersect and their union coincide with $\mathcal{C}_0$. Following Wiener (cf [45]), we associate with each such quasi-interval, say $I_{2^n}(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}})$, a positive number $p\{I_{2^n}(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}})\}$ equal to

$$\int_{\iota_{\ell_1}} \cdots \int_{\iota_{\ell_{2^n}}} \prod_{j=1}^{2^n} u\left(y_j - y_{j-1}, t_j - t_{j-1}\right) dy_1 \cdots dy_{2^n}, \tag{2.4}$$

where u is related to the Gaussian density g trough (1.15) with $x_0 = \mu = 0$. For convenience we put $t_0 = y_0 = 0$. Using property (1.14) we obtain for each positive integer n that

$$\sum_{\ell_1 \cdots \ell_{2^n}=1}^{2^n} p\left\{I_{2^n}\left(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}}\right)\right\} = 1. \tag{2.5}$$

Therefore the number defined by (2.4) is called the *probability* associated with the quasi-interval $I_{2^n}(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}})$.

As is mentioned in section 2.2, each quasi-interval of the n^{th} stage may be represented by the union of certain number of quasi-intervals of the next $n+1^{\text{st}}$ stage. If, say

$$I_{2^n}\left(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}}\right) = \bigcup I_{2^{n+1}}\left(\mathbf{t}_{2^{n+1}}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^{n+1}}}\right)$$

with a certain union, then it is easily verified that

$$p\left\{I_{2^n}\left(\mathbf{t}_{2^n}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^n}}\right)\right\} = \sum p\left\{I_{2^{n+1}}\left(\mathbf{t}_{2^{n+1}}; \iota_{\ell_1}, \ldots, \iota_{\ell_{2^{n+1}}}\right)\right\}, \tag{2.6}$$

where the summation extends over the same set of indices as in the preceding union.

This procedure of ascribing probabilities to quasi-intervals may be nicely characterized by the following mapping. At the first stage map the quasi-intervals (2.1) to the unit interval by starting at the origin and laying out on the α-axis 4 adjoining segments whose lengths are $p\{I_2(\mathbf{t}_2; \ell_1, \ell_1)\}$, $p\{I_2(\mathbf{t}_2; \ell_1, \ell_2)\}$, $p\{I_2(\mathbf{t}_2; \ell_2, \ell_1)\}$ and $p\{I_2(\mathbf{t}_2; \ell_2, \ell_2)\}$, respectively. At the second stage map the quasi-intervals (2.3) to the unit interval by continuing to translate the probabilities into lengths and by arranging the necessary segments in such a way that if a given quasi-interval $I_4(\mathbf{t}_4; \iota_{\ell_1}, \ldots, \iota_{\ell_4})$ is a portion of a certain quasi-interval of the first stage, then the corresponding segments stand in the same relation. It should be clear now that if this procedure is kept up indefinitely, the lengths of the image intervals on $0 \leq \alpha \leq 1$ will tend to zero.

In fact, no specification (2.4) is needed to satisfy the properties (2.5) and (2.6), but the following property, called *equicontinuity*, is based upon this special form of the associated probabilities (Gaussian form as specified in section 1.3). Let us formulate this as a separate proposition. For $h \in (0, \infty)$ and $\lambda \in \left[0, \frac{1}{2}\right)$ denote by $\mathcal{C}_0(h, \lambda)$ the subset of $\mathcal{C}_0$ consisting of functions $f \in \mathcal{C}_0$ such that for all dyadic instants t_1 and t_2

$$|f(t_1) - f(t_2)| \leq h|t_1 - t_2|^\lambda. \tag{2.7}$$

Note that if $h < h'$, then $\mathcal{C}_0(h, \lambda) \subset \mathcal{C}_0(h', \lambda)$. Denote by $\mathcal{C}_0(h, \lambda)^c$ the complement set to $\mathcal{C}_0$.

PROPOSITION 2.1. *For $h \in (0, \infty)$ and $\lambda \in \left[0, \frac{1}{2}\right)$ the subset $\mathcal{C}_0(h, \lambda)^c$ of $\mathcal{C}_0$ can be enclosed in a union of quasi-intervals whose probability (the sum of the probabilities of the involved quasi-intervals) is less than a certain number $p(h)$ which tends to zero, i.e. $p(h) \to 0$ as $h \to \infty$.*

For the present we assume this result, since the proof is too lengthy to be reproduced here, though not technically difficult, see e.g. WIENER et. al. [45], section 2.3. For a general context on equicontinuity sets of functions and related results, including Ascoli's theorem which we are going to apply below, see DIEUDONNÉ [14], section 7.5. In order to use these general arguments we note first that proposition 2.1 allows us to associate with any $\varepsilon > 0$ a positive number $h_0 = h_0(\varepsilon)$ and to confine our considerations to h's such that $p(h) < \varepsilon$ for $h > h_0$. We then delete a denumerable set of quasi-intervals of total probability $< \varepsilon$ so as to obtain the remainder in $\mathcal{C}_0$ that consists of functions satisfying (2.7) for all dyadic couples of instants from $[0, T]$. But we seek more - we want a set of functions satisfying (2.7) at all couples of instants from $[0, T]$, not necessarily dyadic. Therefore we have to modify the set obtained above (of functions f satisfying (2.7) at all dyadic couples of instants), associating with each element f a unique continuous function F by identity $F(t) = f(t)$ at a dyadic t and $F(t) = \lim_{t_n \to t} f(t_n)$ where $\{t_n\}_{n=1,2,\ldots}$ is a sequence of dyadic instants converging to t. This modification yields the set of continuous functions F with property (2.7) at each $t_1, t_2 \in [0, T]$. This is an equicontinuous class of uniformly bounded (by the above h) functions and in virtue of Ascoli's theorem mentioned above every sequence within this class has a uniformly convergent subsequence. The limit is itself continuous, with the equicontinuity property (2.7) at each $t_1, t_2 \in [0, T]$.

In view of the above construction and of proposition 2.1, we are led to shrink the interval $0 \leq \alpha \leq 1$ by removal of a set of points of a negligible total length, so that every point α that remains has a threefold characterization:

a) by a sequence of intervals closing down on it,

b) by the sequence of corresponding quasi-intervals and

c) by a uniquely determined function F common to this sequence of quasi-intervals, equicontinuous in the above sense.

We denote this function by $w(\cdot, \alpha)$. For variable α these are the trajectories of our Brownian motion.

2.4. Quadratic variation

The Brownian paths, although almost all continuous, turn out to be very irregular in nature. It can be shown, for instance, that for almost all α and each fixed instant t

$$\limsup_{\Delta t\to 0} \frac{w(t+\Delta t,\alpha)-w(t,\alpha)}{\Delta t} = \infty.$$

In other words, at each instant t the Brownian paths have infinite upper derivatives. Some explanation of this phenomenon is provided by the fact that in the prelimiting situation our symmetric random walk was allowed to make steps of considerably larger order of magnitude than the length of time intervals (see the discussion at the end of section 1.2).

The next uncommon fact about the irregular character of the Brownian paths is that they almost all have infinite length. This follows from the following general remark concerning any continuous function f of time $t \in [0,T]$: for any grid $\{t_0, t_1, \ldots, t_N\}$

$$\sum_{j=1}^{N} [\Delta f(t_j)]^2 \le \max_{j\in\{1,\ldots,N\}} |\Delta f(t_j)| \sum_{j=1}^{N} |\Delta f(t_j)|$$

with $\Delta f(t_j) = f(t_j) - f(t_{j-1})$. Hence, if the continuous function is of bounded variation (in the sense that the sum on the right is bounded independently of the choice of the grid or, geometrically, that the graph has infinite length) the sum on the left vanishes as the mesh width tends to zero. The following proposition asserts that this sum tends to zero for almost no Brownian path, hence almost no path has bounded variation.

PROPOSITION 2.2. *For any time interval* $[0,t]$*, let* $\{t_0, t_1, \ldots, t_N\}$ *be the dyadic grid of equidistant instants* $t_j = \frac{j}{N}t$ *with* $N = 2^n$*. Then for almost all* α *the sum of squares of the increments converges as* $n \to \infty$*:*

$$\sum_{j=1}^{N} [\Delta w(t_j,\alpha)]^2 \to \sigma^2 t. \tag{2.8}$$

The proof of this important result lays beyond the scope of the present paper (though probabilistic proofs are not very complicated, see e.g. [15], section 8.2, or [26], section 2.2). Instead, let us provide some intuitive explanation by turning back to the prelimiting situation of the symmetric random walk: within each time interval the square of the step size equals to the length of this interval multiplied by σ^2. Hence, the sum of squares yields the length t of the whole interval $[0,t]$ multiplied by the same σ^2.

Note that the right-hand side in (2.8) is the same for almost all α. The limiting function $\sigma^2 t$ is special: within a wide class of stochastic processes (martingales with continuous sample paths, see [41], proposition 7.3) with convergent sums of type (2.8), only Brownian motion possesses this limit. Generally, such a limit, obtained from the increments of a certain underlying process X, depends on α and therefore constitutes a stochastic process with non-decreasing sample paths of bounded variation. It is usually denoted by $\langle X \rangle$ and called the *quadratic variation process* for X. If, for instance, X is a stochastic process of bounded variation, then $\langle X \rangle_t \equiv 0$. As we have already seen $\langle \mathcal{W} \rangle_t = \sigma^2 t$. Regarding quadratic variation processes, this is all we need for the present, see [41] for further references. In addition we only observe, anticipating section 5, that also $\langle \grave{R}^\circ \rangle_t = \sigma^2 t$ for the discounted return process $\grave{R}^\circ = \{\grave{R}^\circ_t\}_{t\in[0,T]}$, since the presence of the drift μt in (5.3) is immaterial.

2.5. Itô's integral

Since almost no Brownian path is of bounded variation, the integral with respect to $\mathcal{W}$ cannot be defined in any conventional way, unless an integrand itself is of bounded variation, as in the latter case it can be defined in the usual Riemann-Stieltjes sense, see e.g. [23], p 55, or [34], pp 14–15. Indeed, if the integrand Φ has sample paths $\phi(\cdot, \alpha)$ of bounded variation for almost all α, then one can overcome difficulties on defining $\int_0^t \Phi_\theta d\mathcal{W}_\theta$ by means of the integrating by parts formula which yields

$$\int_0^t \phi(\theta, \alpha) dw(\theta, \alpha) = \phi(t, \alpha) w(t, \alpha) - \int_0^t w(\theta, \alpha) d\phi(\theta, \alpha). \tag{2.9}$$

This device is efficient for our purposes in section 5, since the integrands are interpreted there as investor's portfolios during a finite trading period that by nature cannot have sample paths of infinite length.

However in theory we need more, for instance, the integral $\int_0^t \mathcal{W}_\theta d\mathcal{W}_\theta$, which cannot be defined in the above sense. An attempt to evaluate this integral explicitly would not lead to usual answer $\frac{1}{2}\mathcal{W}_t^2$ (as would be the case, if $\mathcal{W}$ were of bounded variation). Instead, we get

$$2\int_0^t \mathcal{W}_\theta d\mathcal{W}_\theta = \mathcal{W}_t^2 - \langle \mathcal{W} \rangle_t \tag{2.10}$$

with $\langle \mathcal{W} \rangle_t = \sigma^2 t$ the quadratic variation of $\mathcal{W}$, see the previous section. In order to understand how the additional term (referred sometimes to as Itô's correction term) enters into consideration, look at the following Riemann-Stieltjes sum for this integral: with the notations of proposition 2.2

$$2\sum_{j=1}^N w(t_{j-1}, \alpha) \Delta w(t_j, \alpha) = w^2(t, \alpha) - \sum_{j=1}^N [\Delta w(t_j, \alpha)]^2.$$

The identity is obtained by elementary algebra, cf [23], p 60, [26], p 157. Let now $N \to \infty$. According to Itô's integration theory, the left hand side tends to $2\int_0^t w(\theta, \alpha) dw(\theta, \alpha)$, while the second term on the right tends to $\sigma^2 t$ by proposition 2.2.

Surely, it is just impossible to enter here in details of stochastic calculus. For a good introduction we have already referred to [41] where further references can be found. For the same purposes one can also use [34], section 1.1.3, [23], chapter 4, [26], section 4.5 (or [37] for more advanced theory). In the sequel we only intend to give some insight in formulas needed in section 5. Firstly, we observe that the work on integration begins with a class of elementary integrands so that there is no confusion about how to integrate them. These are so-called simple processes whose sample paths are piecewise constant functions with discontinuities at certain fixed instants. They depend at each $t \in [0,T]$ only on the past of Brownian motion. See [41], formula (5.2). Then the integral with respect to $\mathcal{W}$ over any interval $[0,t]$ is defined in an obvious manner as the corresponding Riemann-Stieltjes sum: if Φ is an integrand whose sample path is observed to jump at instant t_j with $\phi_j(\alpha), j = 0,1,\ldots,n$, then $\int_0^t \Phi_\theta d\mathcal{W}_\theta$ is defined by

$$\int_0^t \phi(\theta,\alpha)dw(\theta,\alpha) = \sum_{j=0}^{n} \phi_j(\alpha)[w(t_{j+1}\wedge t,\alpha) - w(t_j \wedge t,\alpha)],$$

cf [41], formula (5.3); for properties of this integral, see [41], proposition 5.2. We have, for instance, that the quadratic variation of the continuous process $X_t = \int_0^t \Phi_\theta d\mathcal{W}_\theta$ is

$$\langle X\rangle_t = \int_0^t \Phi_\theta^2 d\langle \mathcal{W}\rangle_\theta = \sigma^2 \int_0^t \Phi_\theta^2 d\theta. \tag{2.11}$$

Next, the definition is extended to a class of integrands approachable in a certain sense by sequences of simple processes. The integral of an integrand from this class is defined as the limit of the corresponding integrals of simple processes (the limit turns out to be independent of a particular choice of the approximating sequence). This is quite a delicate task. What makes theory operational is that one can take care about integration rules known from ordinary calculus. For instance, the chain rule says that if X is as above and Ψ is a suitable integrand, then

$$\int_0^t \Psi_\theta dX_\theta = \int_0^t \Psi_\theta \Phi_\theta d\mathcal{W}_\theta. \tag{2.12}$$

Next another important rule – Itô's formula (called sometimes the change of variables rule). Let u be a certain sufficiently smooth function of its arguments x and t, a space coordinate and time respectively, at least continuously differentiable in time and twice differentiable in space. Then with X possessing property (2.11) and with the notations (1.18) we have

$$\begin{aligned} u(X_t,t) &= u(X_0,0) + \int_0^t u_x(X_\theta,\theta)dX_\theta + \int_0^t u_t(X_\theta,\theta)d\theta \\ &+ \frac{1}{2}\sigma^2 \int_0^t u_{xx}(X_\theta,\theta)d\langle X\rangle_\theta. \end{aligned} \tag{2.13}$$

The proof is based on the fact that the quadratic variation $\langle X\rangle$ is non-zero, given by (2.11). Hence, when the function $u(\cdot,\cdot)$ is developed in Taylor's expansion, there is a (unconventional) contribution from the second order term that yields Itô's correction - the last term in (2.13). This is a fundamental formula of stochastic calculus and its proof can be found in the textbooks we refer to, e.g. in [34], proposition 1.1.4. This section is closed by two applications.

(i) Put $u(x,t) = x^2$. From (2.13) we obtain an extension of (2.10) to X.

(ii) Put $u(x,t) = e^{x+\mu t}$ with some constant μ and apply (2.13) to $X = \mathcal{W} - \frac{1}{2}\langle\mathcal{W}\rangle$, as is done in [41], p 371. We obtain that $Z_t = e^{\mathcal{W}_t + \mu t - \frac{1}{2}\sigma^2 t}$ satisfies the following linear integral equation $Z_t = 1 + \int_0^t Z_\theta d\mathcal{W}_\theta$. The solution Z is known as Doléans-Dade (or stochastic) exponential and is denoted by $Z = \mathcal{E}(\mathcal{W})$. Clearly, if $\mathcal{W}$ in this integral equation were an ordinary function of bounded variation, say F, then we would simply have the ordinary exponential $Z = \mathcal{E}(\mathcal{W}) = e^F$.

3. Heat equations

3.1. Fokker-Planck equation

In this section we follow Hida [26] in his short account of Einstein's ideas concerning Brownian motion. We need to consider only the projection of the motion onto a line. The density of pollen grains per unit length at an instant t will be denoted by $u(\cdot, t)$. This is a function of a space coordinate $x \in (-\infty, \infty)$. Suppose that the movement occurs uniformly in both time and space, so that the proportion of the pollen grains moved from x to $x+y$ in a time interval $(t, t+\Delta t)$ of length Δt may be written $p(y, \Delta t)$, as this is independent of x and t. For this time interval we thus obtain the superposition

$$u(x, t+\Delta t) = \int_{-\infty}^{\infty} u(x-y, t) p(y, \Delta t) dy \tag{3.1}$$

(valid under suitable smoothness conditions on the functions u and p, none of our present concern). Further, the function p is supposed to be a probability density possessing property (1.14), with the first two moments proportional to Δt:

$$\int_{-\infty}^{\infty} y^i p(y, \Delta t) dy = \begin{cases} \mu \Delta t & \text{if } i = 1 \\ \sigma^2 \Delta t & \text{if } i = 2 \end{cases}$$

where the proportionality parameters $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$ are called the *drift* and *diffusion* coefficients, respectively, for a reason to become clear soon. Then the Taylor expansion of (3.1) for small increments Δt

$$\begin{aligned} & u(x,t) + \Delta t\, u_t(x,t) + \cdots \\ = & \int_{-\infty}^{\infty} \{u(x,t) - y u_x(x,t) + \frac{1}{2} y^2 u_{xx}(x,t) - \cdots\} p(y, \Delta t) dy \end{aligned}$$

(recall notations (1.18) for corresponding partial derivatives) reduces to the following second order partial differential equation

$$u_t = \frac{1}{2}\sigma^2 u_{xx} - \mu u_x. \tag{3.2}$$

This is the well-known *Fokker-Planck equation* for diffusion with drift (see e.g. [21], section 14.6 or [8], section 5.6). Suppose that initially the grain is at certain site x_0 say, which yields the initial condition (1.16). Then integrating (3.2), we obtain

PROPOSITION 3.1. *The solution of the partial differential equation (3.2) subject to the initial condition (1.16) is given by (1.15).*

PROOF. To verify that u given by (1.15) satisfies (3.2), make use of $g'(x) = -xg(x)$ and calculate directly from (1.15) the corresponding partial derivatives (1.18). As was already seen in section 1.3 the initial condition (1.16) is satisfied as well. The proof is complete. □

According to the theory of stochastic processes in the particular case of $x_0 = 0$, $\mu = 0$ and $\sigma = 1$ the u thus obtained turns out to be the transition probability function for standard Brownian motion (viewed as a Markov process, see e.g. [26], section 2.4). In general this is the transition probability function for *diffusion* that consists of two terms, a deterministic term plus a stochastic term. The latter term is Brownian motion (which starts from an arbitrary site x_0, not necessarily the origin) scaled by σ, a constant associated with the medium. The deterministic term occurs only in presence of external field of source (e.g. gravity) which causes the drift μt.

In the next section we will discuss the strong similarity between diffusion and the molecular mechanism of thermal conductance. Of course, in contrast to diffusion there are no actual migration particles bearing heat, so that in this case relevant partial differential equations of type (3.2) have to be derived by phenomenological considerations.

3.2. Thermal conductance

In physics equation (3.2) emerges again in the following problem of thermal conductance. Let u be a certain sufficiently smooth function of its arguments x and t, space and time coordinates respectively, at least continuously differentiable in time and twice differentiable in space. Consider the partial differential equation (3.2) with $\mu = 0$. In the present context, this is called the *heat equation*, because it describes the temperature distribution of a certain homogeneous isotropic body, in the absence of any heat sources within the body (it is enough for our purposes to restrict the consideration to special case of 'scalar body', a rod; see e.g. [6], [22], [29] or [42]). Otherwise, if a certain heat source causes temperature change proportional to time, with a proportionality parameter μ, then the heat equation is (3.2). In the present context the parameter σ^2 is called *thermal diffusivity* (as was pointed out by Einstein, $\sigma^2 = \frac{RT}{Nf}$ where R is universal constant depending on suspending material, T the absolute temperature, N the Avogadro number and f the coefficient of friction, see e.g. [45], section 2.1).

Furthermore, in cases where the heat flow consists of both conduction and radiation, the heat equation has to be altered to include the effects of radiation. For instance, consider the temperature distribution in a rod which has a longitudinal heat flow due to conduction, as well as heat radiation from the surface. By Newton's law of surface heat transfer the rate of cooling per unit length of bar can be estimated by $r\bar{u}(\cdot,\cdot)$ where r is a positive constant and $\bar{u}$ a function of x and t expressing the excess of temperature of the bar over its surroundings. The heat equation then becomes

$$\bar{u}_t + r\bar{u} = \frac{1}{2}\sigma^2\bar{u}_{xx} - \mu\bar{u}_x. \tag{3.3}$$

But the latter equation is easily reduced to the previous (3.2) by a change of variable

$$\bar{u}(x,t) = e^{-rt}u(x,t) \tag{3.4}$$

with u satisfying (3.2). Thus, by integrating (3.2) and substituting the result into (3.4) we get the solution to (3.3). If, for instance, we look for a particular solution of (3.3) which yields the temperature distribution within the body due to a unit source of heat that at the initial date $t = 0$ is totally concentrated at a site x_0, then according to proposition 3.1 the solution to (3.3) is given by

$$\bar{u}(x,t) = e^{-rt}g(x; x_0 + \mu t, \sigma^2 t). \tag{3.5}$$

Consider now the situation in which the distribution of temperature throughout the body at the initial date $t = 0$ is not concentrated at a certain site x_0 as before, but presented by a sufficiently smooth function f. Then the problem of thermal conductance consists of integrating the heat equation (3.3) subject to the initial condition

$$\bar{u}(x,0) = f(x). \tag{3.6}$$

PROPOSITION 3.2. *The solution of the heat equation (3.3) subject to the initial condition (3.6) is given by*

$$\hat{v}(x,t) = \int_{-\infty}^{\infty} \bar{u}(y-x,t)f(y)dy \tag{3.7}$$

with $\bar{u}$ of form (3.5) where $x_0 = 0$.

PROOF. As was already noted, one can set $r = 0$ without loss of generality. Then it is easily seen that (3.7) satisfies (3.2) since the required partial differentiation can be carried out under the integral sign. To complete the proof, evaluate (3.7) at $t = 0$ by taking into consideration (1.15) and the property (1.17) of the limit (1.16). □

As was mentioned in section 1.1, the option valuation problem of section 5.3 reveals strong similarity to the problem of heat conduction where at a certain instant T (call it *maturity*, as usual) the distribution of temperature throughout the body is given and it is required to restore the previous process.

Suppose first that at maturity $t = T$ the temperature distribution is totally concentrated at a certain site x_0. Then the problem consists in solving equation

$$-\bar{u}_t + r\bar{u} = \frac{1}{2}\sigma^2\bar{u}_{xx} - \mu\bar{u}_x \tag{3.8}$$

subject to the boundary condition $\bar{u}(x,T) = \delta(x-x_0)$. By the same arguments as above, we get the solution (3.5) but now with time to maturity $\tau = T - t$ on the right hand side instead of t.

Finally, let the temperature distribution at maturity $t = T$ be presented by a certain sufficiently smooth function f, so that the boundary condition becomes

$$\bar{u}(x,T) = f(x). \tag{3.9}$$

Then proposition 3.2 yields

COROLLARY 3.3. *The solution of the heat equation (3.8) subject to the boundary condition (3.9) is given by*

$$\hat{u}(x,t) = \int_{-\infty}^{\infty} \bar{u}(x-y;\tau)f(y)dy \tag{3.10}$$

with time to maturity $\tau = T - t$ and with $\bar{u}$ of form (3.5) where $x_0 = 0$.

3.3. Modified heat equation

The problem considered at the end of the previous section is a special case of the general *Cauchy problem*, the solution of which is known as the *Feynman–Kac formula*, see e.g. [29], section 5.7. In this section another special case will be treated, the solution of which yields in section 5 the celebrated *Black–Scholes* formula (5.20) for option valuation, cf [3], [24] or [29], section 5.8.

Let h be a function of a positive space coordinate $x \in (0,\infty)$ and time $t \in (0,\infty)$. Consider the following partial differential equation for x and t

$$-h_t + rh = \frac{1}{2}\sigma^2 x^2 h_{xx} - \mu x h_x. \tag{3.11}$$

First look for the particular solution of this equation subject to the boundary condition that for sufficiently smooth functions f of a positive argument we have at any positive site x_0

$$\lim_{t\uparrow T}\int_0^{\infty} h(\frac{x}{x_0},t)f(x)\frac{dx}{x} \to f(x_0). \tag{3.12}$$

It is easily verified by the same arguments as above that the solution is given by $h(x,t) = \bar{u}(\log x,\tau)$ where $\bar{u}$ is again given by (3.5) but this time with $x_0 = 0$ and with $\mu + \frac{1}{2}\sigma^2$ instead of μ, i.e.

$$h(x,t) = e^{-r\tau}g(\log x; \mu\tau + \frac{1}{2}\sigma^2\tau, \sigma^2\tau). \tag{3.13}$$

Indeed, (3.11) is obtained by differentiating $h(x,t) = \bar{u}(\log x, \tau)$ and taking (3.8) into consideration, and (3.12) by taking the limit as $\tau \to 0$ on the right-hand side of

$$\int_0^\infty h(\frac{x}{x_0}, t) f(x) \frac{dx}{x} = e^{-r\tau} \int_{-\infty}^\infty g(y; \mu\tau + \frac{1}{2}\sigma^2\tau, \sigma^2\tau) f(x_0 e^y) dy.$$

We apply this to the following boundary problem. Let H be a sufficiently smooth function of a positive argument (e.g. $(x-K)^+$ as in section 4.5). Integrate (3.11) subject to the boundary condition

$$\bar{h}(x,T) = H(x). \tag{3.14}$$

PROPOSITION 3.4. *The solution of the modified heat equation (3.11) subject to the boundary condition (3.14) is given by*

$$\bar{h}(x,t) = \int_0^\infty h\left(\frac{x}{y}, t\right) H(y) \frac{dy}{y} \tag{3.15}$$

with h defined by (3.13).

PROOF. Write (3.15) in the form

$$\bar{h}(x,t) = \int_{-\infty}^\infty \bar{u}(\log x - y, \tau) H(e^y) dy \tag{3.16}$$

and apply the same arguments as in the course of proving proposition 3.2. □

By obvious change of variables we get from (3.13) and (3.16) that

$$\bar{h}(x,t) = e^{-r\tau} \int_{-\infty}^\infty g(y, -\mu\tau, \sigma^2\tau) H(x e^{y - \frac{1}{2}\sigma^2\tau}) dy, \tag{3.17}$$

the formula to which we will return in the sections 4.5 and 5.

4. BINARY MODEL

4.1. Self-financing strategies

In this section we briefly review the theory of binary securities markets developed in the previous parts I and II, section 3. Suppose that one invests an amount $v \geq 0$ in two assets described in section 1.2. Let $\pi^N = \{\Psi_t^N, \Phi_t^N\}_{t \in [0,T]}$ be a trading strategy of the investor, with the bond and stock components given by formulas (3.2.2) and (3.2.3) in part II. At each instant $t \geq 0$ the investor holds Ψ_t^N shares of the bond and Φ_t^N shares of the stock so that the market value of the holding is

$$V^N(t;\pi) = \Psi_t^N B_t^N + \Phi_t^N S_t^N \tag{4.1}$$

with the initial value determined by the endowment $v = V^N(0;\pi)$. The process $V^N(\pi) = \{V^N(t;\pi)\}_{t \in [0,T]}$ is called the *value process* for a strategy π^N.

Definition. A trading strategy is said to be *self-financing* if the construction is founded only on the initial endowment so that all changes in the portfolio values are due to capital gains during trading and no infusion or withdrawal of funds takes place. Then the corresponding portfolio satisfies the condition: for all $t \in [0, T]$

$$B_-^N \cdot \Psi_t^N + S_-^N \cdot \Phi_t^N = 0.$$

Recall that under the self-financing condition we have at $t \in (0, T]$ not only (4.1) but also

$$V^N(t-; \pi) = \Psi_t^N B_{t-}^N + \Phi_t^N S_{t-}^N, \tag{4.2}$$

cf. part I, formula (3.2.3). Using the integrating by parts formula we get

Proposition 4.1. *A trading strategy π^N is self-financing if and only if its discounted value process $\grave{V}^N(\pi) = \{\grave{V}_t^N(\pi)\}_{t\in[0,T]}$ admits the following integral representation: at each $t \in [0, T]$*

$$\grave{V}^N(t; \pi) = v + \Phi^N \cdot \grave{S}_t^N. \tag{4.3}$$

Proof. See part II, proposition 3.2.2. □

In order to rewrite the integral representation 4.3 in Clark's form (cf. part II, proposition 3.2.6; for Clark's representation, see [35] or [34], proposition 1.3.5), we recall definition 3.2.5 in part II of the difference operator D in the state space: the process $DS^N = \{DS_t^N\}_{t\in[0,T]}$ is defined so that $DS^N(0) = DS_1^N$ and $DS^N(t) = DS^N_{\lceil \frac{tN}{T} \rceil}$ with DS_n^N, $n \in \{1, \ldots, N\}$ in one of the states $\{D_k(S_n^N)\}_{k=1,\ldots,2^{n-1}}$ where $D_k(S_n^N) = s_{2k,n}^N - s_{2k-1,n}^N$. For instance, if $t \in \left(\frac{(n-1)T}{N}, \frac{nT}{N}\right]$ and if $S^N(t-)$ is in state $s_{k,n-1}^N$, then $DS^N(t)$ is in state $D_k(S_n^N)$. Note that in view of (1.5)

$$\frac{D_k(S_n^N)}{2\sigma\sqrt{\Delta t_n}} = \frac{s_{k,n-1}^N \left(r_{2k,n}^N - r_{2k-1,n}^N\right)}{2\sigma\sqrt{\Delta t_n}} = s_{k,n-1}^N(1 + \xi_{k,n-1}^N) \tag{4.4}$$

with

$$\xi_{k,n-1}^N = \frac{a + b + \eta_{2k,n}^N + \eta_{2k-1,n}^N}{2\sigma}\sqrt{\Delta t_n}, \tag{4.5}$$

which under condition S is a negligible remainder term.

We now turn to the value process for a self-financing strategy. According to part II, remark 3.2.3, this is a process of the same type as the stock price process, since $\grave{V}^N(t; \pi) = \grave{V}^N_{\lfloor tN/T \rfloor}(\pi)$ where each $\grave{V}_n^N(\pi)$, $n \in \{1, \ldots, N\}$, may occupy one of the states

$$\grave{v}_{kn}^N(\pi) = v + \sum_{\nu=1}^{n} \Phi_\nu^N(s_{k_{\nu-1},\nu-1}^N)(\grave{s}_{k_\nu,\nu}^N - \grave{s}_{k_{\nu-1},\nu-1}^N)$$

with $k = 1, \ldots, 2^n$, see part II for remark 3.2.3. Next, we let the difference operator D act on V^N in the exactly same manner as on S^N but this time with the states v_{kn}^N in place of s_{kn}^N. Then proposition 3.2.6 in part II tells us that if the process $\frac{DV^N(\pi)}{DS^N} = \{\frac{DV^N(t;\pi)}{DS^N(t)}\}_{t\in[0,T]}$ is formed with the help of the sequence $\{\frac{DV_n^N(\pi)}{DS_n^N}\}_{n=1,\ldots,N}$ by utilizing $\lfloor \frac{tN}{T} \rfloor$ as before, the integrand in (4.3) can be identified as follows: $\Phi^N(t) = \frac{DV^N(t;\pi)}{DS^N(t)}$, i.e.

$$\grave{V}^N(t;\pi) = v + \frac{DV^N(\pi)}{DS^N} \cdot \grave{S}_t^N. \tag{4.6}$$

This is called Clark's formula, cf. part I, section 3.3 or part II, proposition 4.2.2.

4.2. Risk neutral probabilities

Let us focus our attention on markets excluding arbitrage opportunities (see part I, section 6). Then for each $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$, the numerical values of *risk neutral probabilities*

$$p_{2k,n}^N = \frac{\grave{s}_{k,n-1}^N - \grave{s}_{2k-1,n}^N}{\grave{s}_{2k,n}^N - \grave{s}_{2k-1,n}^N} = \frac{\rho_n^N - r_{2k-1,n}^N}{r_{2k,n}^N - r_{2k-1,n}^N} \tag{4.7}$$

(cf (1.1) and (1.4)) are positive and $p_{2k-1,n}^N = 1 - p_{2k,n}^N$. As was emphasized in part II, section 3.2, the usage of this term stems from the fact that every state $s_{k,n-1}^N$ at the trading time t_{n-1} is expressed as a convex combination of two alternative states $\grave{s}_{2k-1,n}^N$ and $\grave{s}_{2k,n}^N$ at the next trading time t_n, weighted by $p_{2k-1,n}^N$ and $p_{2k,n}^N$, respectively. The same is true concerning the states of the value process $V^N(\pi)$ for any self-financing strategy π^N: for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$

$$\grave{v}_{k,n-1}^N(\pi) = p_{2k,n}^N \grave{v}_{2k,n}^N(\pi) + p_{2k-1,n}^N \grave{v}_{2k-1,n}^N(\pi). \tag{4.8}$$

It will be shown in the next lemma that asymptotically the risk neutral probabilities of the upward and downward displacements are equal to $\frac{1}{2}$.

LEMMA 4.2. *Under the conditions B and S the risk neutral probabilities $\{p_{kn}^N\}_{k=1,\ldots,2^n}$ for $n = 1, \ldots, N$, given by (4.7), are approximated as follows*

$$p_{kn}^N \sim \frac{1}{2}\left(1 + (-1)^{k-1}\frac{\mu}{\sigma}\sqrt{\Delta t_n}\right) \tag{4.9}$$

with the parameters $\sigma > 0$ as in (1.5) and μ as in (1.10).

PROOF. For $n = 1, \ldots, N$ let $\{\gamma_{kn}^N\}_{k=1,\ldots,2^n}$ be defined by the equality

$$(1 + \xi_{k,n-1}^N)\gamma_{k,n-1}^N = \varrho_n^N + \mu\xi_{k,n-1}^N - \frac{\eta_{2k,n}^N - \eta_{2k-1,n}^N}{2} \tag{4.10}$$

with r and ϱ_n^N as in condition B, σ and $\{\eta_{kn}^N\}_{k=1,\ldots,2^n}$ as in condition S and $\{\xi_{kn}^N\}_{k=1,\ldots,2^n}$ as in (4.5). Hence under the conditions B and S all of $\{\gamma_{kn}^N\}_{k=1,\ldots,2^n}$ are negligible so that the desired assertion is a consequence of the following identity

$$p_{kn}^N = \frac{1}{2}\left(1 + (-1)^{k-1}\left(\frac{\mu - \gamma_{k,n-1}^N}{\sigma}\right)\sqrt{\Delta t_n}\right) \tag{4.11}$$

which will be proved next.

Since by (1.2) and (1.5)

$$\rho_n^N - r_{2k-1,n}^N = \sigma\sqrt{\Delta t_n} + \left(r + b + \varrho_n^N + \eta_{2k-1,n}^N\right)\Delta t_n$$

and by (1.5) and (4.5)

$$r_{2k,n}^N - r_{2k-1,n}^N = 2\sigma(1 + \xi_{k,n-1}^N)\sqrt{\Delta t_n},$$

it follows from (4.7) that

$$\begin{aligned} \sigma(1 + \xi_{k,n-1}^N)\left(1 - 2p_{2k,n}^N\right) &= \sigma\xi_{k,n-1}^N - \left(r + b + \varrho_n^N + \eta_{2k-1,n}^N\right)\sqrt{\Delta t_n} \\ &= \left(\mu - \varrho_n^N + \frac{\eta_{2k,n}^N - \eta_{2k-1,n}^N}{2}\right)\sqrt{\Delta t_n}. \end{aligned}$$

Thus

$$p_{2k,n}^N = \frac{1}{2}\left(1 - \frac{\mu - \varrho_n^N + \frac{\eta_{2k,n}^N - \eta_{2k-1,n}^N}{2}}{\sigma\left(1 + \xi_{k,n-1}^N\right)}\sqrt{\Delta t_n}\right)$$

which is equivalent to (4.11), since

$$\mu - \varrho_n^N + \frac{\eta_{2k,n}^N - \eta_{2k-1,n}^N}{2} = \left(1 + \xi_{k,n-1}^N\right)\left(\mu - \gamma_{k,n-1}^N\right)$$

by (4.10). □

4.3. Heat equation in finite differences

It will be shown in this section that under conditions B and S we have the identity in finite differences which has the form of the modified heat equation (3.11), up to certain negligible remainder terms. This identity concerns the states $v_{kn} = v_{kn}^N(\pi)$ of the value process $V = V^N(\pi)$ for a self-financing strategy π. For simplicity, the argument π is suppressed, as well as all the upper indices N. In section 4.1 we already came across first differences $D_k(V_n) = v_{2k,n} - v_{2k-1,n}$ for $n = 1,\ldots,N$ and $k = 1,\ldots,2^{n-1}$. Along with this, we will need to recall definition 2.4.2 in part I of second differences

$$D_k^2(V_n) = v_{4k,n} - v_{4k-1,n} - v_{4k-2,n} + v_{4k-3,n} \tag{4.12}$$

for $n = 2,\ldots,N$ and $k = 1,\ldots,2^{n-2}$.

PROPOSITION 4.3. *Let the conditions B and S hold. Let π be a self-financing strategy and $V = \{V_n\}_{n=0,1,\dots,N}$ its value process. Then for each $n = 2,\dots,N$ and $k = 1,\dots,2^{n-2}$ the following identity hold:*

$$\begin{aligned} &\tfrac{1}{2} \quad (\sigma s_{k,n-2})^2 \frac{D_k^2(V_n)}{D_k(S_{n-1})^2} (1+\xi_{k,n-2})^2 (1+\varsigma_{k,n-2}) \\ &- \quad (\mu + \mu_{k,n-2})\, s_{k,n-2} \frac{D_k(V_{n-1})}{D_k(S_{n-1})} \\ &= \quad -\frac{v_{4k-1,n} - v_{k,n-2}}{t_n - t_{n-2}} + (r+\delta_{n-2})\, v_{k,n-2} + \delta_{k,n-2} \end{aligned} \tag{4.13}$$

where δ_n, $\{\delta_{kn}\}_{k=1,\dots,2^n}$, $\{\varsigma_{kn}\}_{k=1,\dots,2^n}$ and $\{\mu_{kn}\}_{k=1,\dots,2^n}$ are negligible remainder terms, alike $\{\xi_{kn}\}_{k=1,\dots,2^n}$ given by (4.5).

PROOF. We will prove that the identity (4.13) is satisfied with the remainder terms

$$\varsigma_{k,n-2} = 4p_{4k,n}p_{4k-3,n} - 1, \tag{4.14}$$

$$\mu_{k,n-2} = (1+\rho_n)(1+\xi_{k,n-2})\left(\mu - \frac{\gamma_{k,n-2} + \gamma_{2k,n-1}}{2}\right) - \mu, \tag{4.15}$$

$$2\delta_{n-2} = \varrho_{n-1} + \varrho_n + (r+\varrho_{n-1})(r+\varrho_n)\Delta t_n \tag{4.16}$$

and

$$\begin{aligned} \delta_{k,n-2} \quad &= \quad p_{4k,n}((1+\xi_{k,n-1})\, s_{2k,n-1} \frac{D_{2k}(V_n)}{D_{2k}(S_n)} \frac{\gamma_{2k,n-1} - \gamma_{2k-1,n-1}}{2} \\ &+ \quad \frac{v_{4k-1,n} - v_{k,n-2}}{t_n - t_{n-2}} - \frac{v_{4k-2,n} - v_{k,n-2}}{t_n - t_{n-2}}). \end{aligned} \tag{4.17}$$

This will imply the desired assertion, since the remainder terms, given by (4.14) - (4.17) are negligible under the conditions B and S. Indeed, the terms $\{\varsigma_{kn}\}_{k=1,\dots,2^n}$ are negligible due to (4.9), the terms $\{\mu_{kn}\}_{k=1,\dots,2^n}$ due to (1.2), (4.5) and (4.10) and the terms $\{\delta_{kn}\}_{k=1,\dots,2^n}$ due to (4.10) and to the fact that the difference of the second and third terms on the right-hand side is negligible. Finally, the negligibility of δ_n is obvious.

To prove (4.13), we proceed as follows. It will be shown first that (4.13) is equivalent to (4.19) given below. Then the latter identity will be proved. Let us examine (4.13) term by term. Due to (4.4) and (4.14), the first term on the left hand side may be written in the form

$$p_{4k,n}p_{4k-3,n}\frac{D_k^2(V_n)}{2\Delta t_n}.$$

The second term may be written in the form

$$(1+\rho_n)\,\frac{1 - p_{2k,n-1} - p_{4k,n}}{2\Delta t_n} D_k(V_{n-1}),$$

since by (4.11) and (4.15)

$$\begin{aligned}
& (1+\rho_n)\,\sigma\frac{1-p_{2k,n-1}-p_{4k,n}}{\sqrt{\Delta t_n}} \\
= \quad & (1+\rho_n)\left(\mu-\frac{\gamma_{k,n-2}+\gamma_{2k,n-1}}{2}\right) \\
= \quad & \frac{\mu+\mu_{k,n-2}}{1+\xi_{k,n-2}}.
\end{aligned}$$

Next, let us handle the right-hand side of (4.13). The sum of the first two terms on this side equals

$$\frac{(1+\rho_n)\,(1+\rho_{n-1})\,v_{k,n-2}-v_{4k-1,n}}{2\Delta t_n}$$

by (1.2) and (4.16). On verifying this take into consideration that $\Delta t_n = \Delta t_{n-1} = \frac{T}{N}$. Finally, by the same assumptions and by (4.4) the third term on the right, given by (4.17), may be written in the form

$$\frac{p_{4k,n}}{2\Delta t_n}\left(v_{4k-1,n}-v_{4k-2,n}+D_{2k}(V_n)\sqrt{\Delta t_n}\frac{\gamma_{2k,n-1}-\gamma_{2k-1,n-1}}{2\sigma}\right).$$

Thus we obtain the following identity, equivalent to (4.13):

$$\begin{aligned}
& p_{4k,n}p_{4k-3,n}D_k^2(V_n)+(1+\rho_n)\,(p_{2k,n-1}+p_{4k,n}-1)\,D_k(V_{n-1}) \\
= \quad & (1+\rho_n)\,(1+\rho_{n-1})\,v_{k,n-2}-v_{4k-1,n} \\
+ \quad & p_{4k,n}\left(v_{4k-1,n}-v_{4k-2,n}+D_{2k}(V_n)\sqrt{\Delta t_n}\frac{\gamma_{2k,n-1}-\gamma_{2k-1,n-1}}{2\sigma}\right).
\end{aligned}$$

The latter identity, in turn, is equivalent to

$$\begin{aligned}
& (1+\rho_n)\{v_{2k,n-1}p_{4k-3,n}+v_{2k-1,n-1}p_{4k,n} && (4.18)\\
& +(p_{2k,n-1}+p_{4k,n}-1)D_k(V_{n-1})\} \\
- \quad & (v_{4k-1,n}p_{4k-3,n}+v_{4k-2,n}p_{4k,n}) \\
= \quad & (1+\rho_n)(1+\rho_{n-1})v_{k,n-2}-v_{4k-1,n} \\
+ \quad & p_{4k,n}\{v_{4k-1,n}-v_{4k-2,n}+(p_{4k,n}-p_{4k-2,n})D_{2k}(V_n)\} && (4.19)
\end{aligned}$$

in virtue of the following two identities. Firstly

$$\sqrt{\Delta t_n}\frac{\gamma_{2k,n-1}-\gamma_{2k-1,n-1}}{2\sigma}=p_{4k,n}-p_{4k-2,n}$$

which is a consequence of (4.11). Secondly, due to

$$\begin{aligned}
p_{4k,n}p_{4k-3,n}D_k^2(V_n) \quad = \quad & (1+\rho_n)\,(v_{2k,n-1}p_{4k-3,n}+v_{2k-1,n-1}p_{4k,n}) \\
- \quad & (v_{4k-1,n}p_{4k-3,n}+v_{4k-2,n}p_{4k,n})
\end{aligned}$$

which is a consequence of (4.12) and (4.8), since

$$D_k^2(V_n)=\frac{(1+\rho_n)\,v_{2k,n-1}-v_{4k-1,n}}{p_{4k,n}}-\frac{v_{4k-2,n}-(1+\rho_n)\,v_{2k-1,n-1}}{p_{4k-3,n}}.$$

Thus the equivalence of (4.13) and (4.19) is proved. It remains to prove (4.19). By (4.8) in its non-discounted form

$$\begin{aligned} & v_{2k,n-1}p_{4k-3,n} + v_{2k-1,n-1}p_{4k,n} - (1+\rho_{n-1})\, v_{k,n-2} \\ = \; & v_{2k,n-1}\,(p_{4k-3,n} - p_{2k,n-1}) + v_{2k-1,n-1}\,(p_{4k,n} - p_{2k-1,n-1}) \\ = \; & (v_{2k,n-1} - v_{2k-1,n-1})\,(1 - p_{2k,n-1} - p_{4k,n}) \\ & + v_{2k,n-1}\,(p_{4k,n} - p_{4k-2,n}) \end{aligned}$$

and

$$\begin{aligned} & v_{4k-1,n}p_{4k-3,n} + v_{4k-2,n}p_{4k,n} - v_{4k-1,n} \\ = \; & v_{4k-2,n}p_{4k,n} - v_{4k-1,n}p_{4k-2,n} \\ = \; & v_{4k-1,n}\,(p_{4k,n} - p_{4k-2,n}) - (v_{4k-1,n} - v_{4k-2,n})\, p_{4k,n}. \end{aligned}$$

These two identities imply (4.19). The proof is complete. □

4.4. Completeness, hedging strategy and option valuation

Given the states of the discounted stock price over the entire trading period $[0,T]$, the risk neutral probabilities $\{p^N_{kn}\}_{k=1,\dots,2^n}$ are determined by (4.7). Fix $n \in \{1,\dots,N\}$ and $k \in \{1,\dots,2^N\}$. In the previous parts (part I, section 4.4, and part II, section 3.3) there has been defined the set of weights $\{P^N_{n|\nu}(k)\}_{\nu=0,1,\dots,n}$ with $P^N_{n|\nu}(k) = p^N_{k_{\nu+1},\nu+1}\cdots p^N_{k_n,n}$ for $\nu < n$ and $P^N_{n|n}(k) = 1$. We have also used the notation $P^N_{kn} = P^N_{n|0}(k) = p^N_{k_1,1}\cdots p^N_{k_n,n}$. As before $k_n = \lceil \frac{k-1}{2^{N-n}} \rceil$. The aim was to describe the solution of the system of recurrent equations (cf (4.8))

$$\dot{x}_{k,n-1} = p^N_{2k,n}\dot{x}_{2k,n} + p^N_{2k-1,n}\dot{x}_{2k-1,n} \tag{4.20}$$

for $n = 1,\dots,N$ and $k = 1,\dots,2^{n-1}$, subject to the boundary conditions

$$\dot{x}_{kN} = \dot{w}^N_k(T) = \frac{W(s^N_k(T))}{B^N(T)}, \;\; k = 1,\dots,2^N, \tag{4.21}$$

where $\{\dot{w}^N_k(T)\}_{k=1,\dots,2^N}$ are certain numbers given in the form of a function W of the stock price at maturity, discounted by $B^N(T)$. It will be clear soon why we need to solve this boundary problem. For $n = 0,1,\dots,N$ the solutions $\{\dot{w}_{kn}\}_{k=1,\dots,2^n}$ are obtained by

$$\dot{w}^N_{kn} = \sum_{2^{N-n}(k-1)<j\le 2^{N-n}k} P^N_{N|n}(j)\dot{w}^N_j(T), \;\; k = 1,\dots,2^n. \tag{4.22}$$

In particular

$$w^N_{10} = \sum_{j=1}^{2^N} P^N_j(T)\dot{w}^N_j(T) \tag{4.23}$$

where $P_k^N(T) = P_{kN}^N$. In the trivial case of $W(x) = x$, for instance, the boundary conditions (4.21) are specified by $\grave{x}_{kN} = \grave{s}_k^N(T)$ and the solutions (4.22) and (4.23) reduce to (3.3.7) and (3.3.8) of part II.

Next, we will use (4.22) and (4.23) to describe the *completeness* of the binary market. To this end, consider an investor who is willing to invest now (at $t = 0$) in the bond and the stock in order to attain at the terminal date T a certain wealth, say $W^N(T)$, by trading over N periods without infusion or withdrawal of funds. Knowing the conditions in the market, i.e. knowing the 2^N possible trajectories of the stock price development up to the terminal date T (which correspond as usual to the states $\{s_k^N(T)\}_{k=1,\dots,2^N}$ of the stock price $S^N(T)$), the investor determines the wealth he desires to attain at the terminal date T by evaluating each of these possibilities. In this way $W^N(T)$ is interpreted as a variable which may occupy one of the 2^N possible states: state $w_k^N(T)$ say, if the stock price is in state $s_k^N(T)$. In other words, $W^N(T)$ is a certain function of $S^N(T)$, i.e. $W^N(T) = W(S^N(T))$, and $w_k^N(T) = W(s_k^N(T))$ for $k = 1, \dots, 2^N$.

Definition. A binary market is *complete* if any desired wealth $W^N(T)$ of the above type is attainable with a certain initial endowment: there is a self-financing strategy π^N whose value process at the terminal date attains the identity $V^N(T;\pi) = W^N(T)$. The necessary initial endowment is then $v = V^N(0;\pi)$.

As is shown in part I, proposition 4.3.3, the present market is indeed complete and, moreover, there exists a unique strategy, called the *hedging strategy* against $W^N(T)$, which attains this wealth. In part I, section 4.3, one can find the detailed construction of such strategy. Here we only note that the procedure is based on the solution of the equations (4.20), subject to the boundary conditions (4.21) with the states of the discounted desired wealth on the right hand side. If $\grave{W}_n^N$ is a variable with the possible states $\{\grave{w}_{kn}^N\}_{k=1,\dots,2^n}$ which are identified with the solutions (4.22), then a process $\grave{W}^N = \{\grave{W}_t^N\}_{t\in[0,T]}$ is formed by $\grave{W}^N(t) = \grave{W}_{\lfloor tN/T \rfloor}^N$. Obviously, it starts from (4.23) and at the terminal date T attains the desired wealth. The hedging strategy against $\grave{W}^N(T)$ is then a unique strategy π^N whose value process $V^N(\pi)$ coincides with the process W^N formed above.

In part I, section 5, formula (4.23) is applied to the following problem of option pricing. Suppose that today, at time $t = 0$, we are going to sign a contract that gives us the right to buy one share of a stock at a specified price K, called the *exercise price*, and at a specified time T, called the *maturity* or *expiration time*. If the stock price $S^N(T)$ is below the exercise price, i.e. $S^N(T) \leq K$, then the contract is worthless to us. On the other hand, if $S^N(T) > K$, we can exercise our option: we can buy one share of the stock at the fixed price K and then sell it immediately in the market for the price $S^N(T)$. Thus this option, called the *European call option*, yields a profit at maturity T equal to $H(S^N(T))$ where H is a special function of the stock price $S^N(T)$ of the form $H(x) = \max\{0, x-K\} = (x-K)^+$. It is called the *payoff function* for the European call option. A contract with some fixed payoff function $H^N(T) = H(S^N(T))$, where $H^N(T)$ is a nonnegative variable with possible states $H(s_k^N(T))$ (not necessarily of form $(x-K)^+$) is called a *contingent claim*. The European call option is thus a special contingent claim with payoff $(x-K)^+$.

Now, how much would we be willing to pay at time $t=0$ for a ticket which gives the right to buy at maturity $t=T$ one share of stock with exercise price K? To put this in another way, what is a fair price to pay at time $t=0$ for the ticket? In order to determine the fair price of a contingent claim, consider the following procedure:

(i) construct the hedging strategy against the contingent claim in question, which duplicates the payoff;

(ii) determine the initial wealth needed for construction in (i);

(iii) equate this initial wealth to the fair price of the contingent claim.

In other words, construct the hedging strategy π^N against the contingent claim with a payoff function $H^N(T)$, whose value process $V^N(\pi)$ coincides with a process that is obtained exactly in the same manner as the process $\grave{W}^N$ by solving the equations (4.20), but now subject to the boundary conditions (4.21) with $\grave{h}_k^N(T) = \frac{H(s_k^N(T))}{B^N(T)}$ instead of $\grave{w}_k^N(T) = \frac{W(s_k^N(T))}{B^N(T)}$. This strategy indeed duplicates the payoff. It requires the initial wealth that yields the fair price $C^N = C(H^N)$ of the contingent claim with the payoff function $H^N(T)$ which amounts to the sum on the right hand side of (4.23) with the same substitution of $\grave{h}_k^N(T)$ in place of $\grave{w}_k^N(T)$. The European call option, in particular, has a special payoff function depending only on the stock price at maturity $t_N = T$ and its fair price is

$$C^N = \sum_{j=1}^{2^N} P_j^N(T)(\grave{s}_j^N(T) - \grave{K})^+. \tag{4.24}$$

In conclusion we want to recall the application of formula (4.24) to option valuation in the special binomial markets by means of the Cox-Ross-Rubinstein formula (5.3.1) in part I. For, we are going to use this particular formula in the next section.

4.5. Towards the Black-Scholes market

In the present section the link is sought between the binary market of the present section 4 and the Black-Scholes market of the next section 5. By using certain heuristic arguments we show that under the conditions B and S the Black-Scholes model can serve as an approximation to the binary model, when the number of trading periods N to increase unboundedly and so the length of each trading period $\frac{T}{N}$ tends to zero. For convenience, we will denote differently the bond and stock prices in the Black-Scholes market, namely by $B^{\circ}(t)$ and $S^{\circ}(t)$, respectively. Trading in the latter market is going on continuously so that $t \in [0,T]$. Concerning the bond, the situation is simple, since condition B means that at each fixed $t \in [0,T]$ the approximation $B^N(t) \sim B^{\circ}(t)$ holds. Actually, the approximate bond price is led to be $B^{\circ}(t) = e^{rt}$, cf (1.3) and (5.1) below. As for a risky asset, the stock, in the present context we will be only able to demonstrate certain aspects of the approximation of the process S^N by S°, for more detailed treatment would carry us too far afield. It will be shown, in particular, how to approximate the fair price of the European call option, see proposition 4.5.

We will use expression (4.11) for the approximate risk neutral probabilities, but suppress the negligible remainder terms $\{\gamma_{kn}^N\}_{k=1,\ldots,2^n}$, as it is not hard to verify that they play no part in the asymptotic considerations below. Thus the approximation (4.9) may be used. Since the right hand side in (1.6) is independent of indices k and n, the situation here is asymptotically similar to that of the homogeneous binomial model. Thus we may use the Cox-Ross-Rubinstein option pricing formula (5.3.1) of part I, a special case of the general option pricing formula (4.24). Upon the substitutions (1.2), (1.6) and (4.9), this yields the first approximation to the option pricing formula:

$$\begin{aligned} C^N \quad &\sim \quad \left(1 + r\frac{T}{N}\right)^{-N} \sum_{j=0}^{N} \binom{N}{j} \frac{1}{2^N} \left(1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{N}}\right)^j \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{N}}\right)^{N-j} \\ &\times \quad H\left(s\left(1 + \sigma\sqrt{\frac{T}{N}} + a\frac{T}{N}\right)^j \left(1 - \sigma\sqrt{\frac{T}{N}} - b\frac{T}{N}\right)^{N-j}\right) \qquad (4.25) \end{aligned}$$

with $H(x) = (x-K)^+$, the payoff function for the European call option. It will be shown below that this approximation can be considerably simplified, see proposition 4.5. This is preceded by the following

LEMMA 4.4. *Fix positive integers n and $j \leq n$. Denote $t_j = j\sqrt{2/n}$ so that $\Delta t_j \equiv \sqrt{2/n}$. Then*
(i) for a nonnegative constant c

$$\left(\frac{1 + \sqrt{\frac{c}{2n}}}{1 - \sqrt{\frac{c}{2n}}}\right)^j \sim e^{\sqrt{c}t_j}$$

and (ii) with g the standard normal density (cf (1.13))

$$\frac{1}{2^{2n}}\binom{2n}{n+j} \sim g(t_j)\Delta t_j. \qquad (4.26)$$

PROOF. (i) It follows from $\log(1+x) \sim x$ that

$$j\left\{\log\left(1 + \sqrt{\frac{c}{2n}}\right) - \log\left(1 - \sqrt{\frac{c}{2n}}\right)\right\} \sim j\sqrt{\frac{2c}{n}} = \sqrt{c}t_j,$$

which yields the desired result.
(ii) Presenting the left hand side as the product of

$$a_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

and

$$b_n = \frac{n(n-1)\cdots(n-j+1)}{(n+j)\cdots(n+1)} = \frac{1}{(1+\frac{j}{n})(1+\frac{j}{n-1})\cdots(1+\frac{j}{n-j+1})}$$

we get to show

$$a_n \sim \frac{1}{\sqrt{\pi n}} = \frac{\Delta t_j}{\sqrt{2\pi}}$$

and

$$\log b_n \sim -\frac{j^2}{n} = -\frac{t_j^2}{2}.$$

The former relation follows from Stirling's formula $n! \sim e^{-n}n^n\sqrt{2\pi n}$, truly from its consequence

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}},$$

see [4], section 1.2. The latter one follows from $\log(1+x) \sim x$, since

$$\log b_n = -\sum_{k=0}^{j-1}\log\left(1+\frac{j}{n-k}\right) \sim -\sum_{k=0}^{j-1}\frac{j}{n-k} \sim -\frac{j^2}{n} = -\frac{t_j^2}{2}.$$

The proof of (i) is complete. □

There are various methods for proving the next proposition presented in textbooks on probability theory. In the sequel we will continue to follow Breiman [4].

Proposition 4.5. *Under conditions B and S the fair price of the European call option with the payoff function $H(x) = (x-K)^+$, is approximated as follows:*

$$C^N \sim \int_{-\infty}^{\infty} g(y;0,\sigma^2 T)\left(se^{y-\frac{1}{2}\sigma^2 T} - \grave{K}\right)^+ dy \tag{4.27}$$

where $\grave{K} = \frac{K}{B^\circ(T)} = e^{-rT}K$ is the discounted exercise price, cf (5.1).

Proof. Without loss of generality we assume that N is even. Put $N = 2n$. Then the summation in (4.25) may be changed to $\{-n,\ldots,n\}$, which yields

$$C^N \sim C_n^n \sum_{j=-n}^{n} g_{jn}c_n^j H\left(sK_n^n k_n^j\right) \tag{4.28}$$

where g_{jn} denotes the left hand side of (4.26) and

$$C_n = \frac{1 - \frac{\mu^2}{\sigma^2}\frac{T}{2n}}{\left(1 + r\frac{T}{2n}\right)^2} \quad , \quad K_n = \left(1 + \sigma\sqrt{\frac{T}{2n}} + a\frac{T}{2n}\right)\left(1 - \sigma\sqrt{\frac{T}{2n}} - b\frac{T}{2n}\right),$$

$$c_n = \frac{1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{2n}}}{1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{2n}}} \quad , \quad k_n = \frac{1 + \sigma\sqrt{\frac{T}{2n}} + a\frac{T}{2n}}{1 - \sigma\sqrt{\frac{T}{2n}} - b\frac{T}{2n}}.$$

By the well-known property of exponentials

$$C_n^n \sim e^{-rT - \frac{1}{2}\frac{\mu^2 T}{\sigma^2}}$$

and

$$K_n^n \sim \left(1 + (a - b - \sigma^2)\frac{T}{2n}\right)^n \sim e^{\frac{a-b}{2}T - \frac{1}{2}\sigma^2 T} = e^{(r+\mu)T - \frac{1}{2}\sigma^2 T},$$

cf (1.10). Apply now lemma 4.4. Assertion (ii) gives an approximation to g_{jn} and assertion (i) gives

$$c_n^j \sim e^{-\frac{\mu\sqrt{T}t_j}{\sigma}} \quad \text{and} \quad k_n^j \sim e^{\sigma\sqrt{T}t_j}.$$

These approximations reduce (4.28) to

$$C^N \sim \frac{e^{-rT}}{\sqrt{2\pi}} \sum_{j=-n}^{n} e^{-\frac{1}{2}\frac{(\sigma\sqrt{T}t_j + \mu T)^2}{\sigma^2 T}} H(se^{\sigma\sqrt{T}t_j + (r+\mu)T - \frac{1}{2}\sigma^2 T})\Delta t_j.$$

Put $\tau_j = \sigma\sqrt{T}t_j$. Then

$$C^N \sim e^{-rT} \sum_{\tau_j \in \mathcal{T}_n} g(\tau_j; -\mu T, \sigma^2 T) H(se^{\tau_j + (r+\mu)T - \frac{1}{2}\sigma^2 T})\Delta\tau_j$$

with $\mathcal{T}_n$ the set $\{j\sigma\sqrt{\frac{2T}{n}}\}_{j=0,\pm1,\ldots,\pm n}$ whose lowest entry $-\sigma\sqrt{2nT}$ tends to $-\infty$ and the largest entry $\sigma\sqrt{2nT}$ to ∞ as $n \to \infty$. So the sum in the latter expression is actually the Riemann sum for the integral

$$\int_{-\infty}^{\infty} g(y + \mu T, 0, \sigma^2 T)\left(se^{y + \mu T - \frac{1}{2}\sigma^2 T} - \grave{K}\right)^+ dy$$

which is independent of μ and equals to the integral on the right-hand side of (4.27). The proof is complete. □

5. Black-Scholes model

5.1. *Assets*

In this section we consider the limiting model for a securities market. According to (1.3), the model for the bond is defined by the linear return process $\mathcal{R}^\circ = \{\mathcal{R}_t^\circ\}_{t\in[0,T]}$ and the exponential price process $B^\circ = \{B_t^\circ\}_{t\in[0,T]}$ with

$$\mathcal{R}_t^\circ = rt \quad \text{and} \quad B_t^\circ = e^{rt} \tag{5.1}$$

where $r > 0$ is a riskless interest rate on the bond. (Note that $B^\circ = \mathcal{E}(\mathcal{R}^\circ)$ in the sense given at the very end of section 2.5.)

The stock is again a risky asset and its return process $R^\circ = \{R_t^\circ\}_{t\in[0,T]}$ is defined in accordance with the right-hand side of (1.6): the cumulative impact up to time t of the terms proportional to $\sqrt{\Delta t_n}$ yields $\mathcal{W}_t$ and that of the terms proportional to Δt_n yields the drift $\frac{1}{2}(a-b)t$, see the discussion at the end of section 1.2. This leads to the following diffusion model $R_t^\circ = \mathcal{W}_t + \frac{1}{2}(a-b)t$. Consequently, the price process on the stock $S^\circ = \{S_t^\circ\}_{t\in[0,T]}$ is now defined by

$$S_t^\circ = s\mathcal{E}(R^\circ)_t = se^{\mathcal{W}_t + \frac{1}{2}(a-b)t - \frac{1}{2}\sigma^2 t} = se^{R_t^\circ - \frac{1}{2}\langle R^\circ\rangle_t} \tag{5.2}$$

(see application (ii) at the end of section 2.5) where $s > 0$ is a fixed current price on the stock $S_0^\circ = s$ and $\langle R^\circ\rangle_t = \sigma^2 t$ as in section 2.4. The discounted stock price process is defined by

$$\grave{S}_t^\circ \equiv \frac{S_t^\circ}{B_t^\circ} = se^{\mathcal{W}_t + \mu t - \frac{1}{2}\sigma^2 t} = se^{\grave{R}_t^\circ - \frac{1}{2}\langle \grave{R}^\circ\rangle_t} \tag{5.3}$$

where $\grave{R}_t^\circ = \mathcal{W}_t + \mu t$ is the corresponding return at instant t, cf (1.10). The relation $\grave{S}^\circ = s\mathcal{E}(\grave{R}^\circ)$ is obtained in section 2.5.

As we know, Brownian motion takes its rise at the origin but afterwards at any consecutive instant $t > 0$ it may visit any site $-\infty < x < \infty$. Accordingly, the non-discounted and discounted stock prices, starting from a fixed state $s > 0$, may occupy at any instant $t \in [0,T]$ and site $-\infty < x < \infty$ one of the states

$$s^\circ(x,t) = se^{xI_{\{t\neq 0\}} + \frac{1}{2}(a-b)t - \frac{1}{2}\sigma^2 t} \quad \text{and} \quad \grave{s}^\circ(x,t) = se^{xI_{\{t\neq 0\}} + \mu t - \frac{1}{2}\sigma^2 t} \tag{5.4}$$

where $I_{\{t\neq 0\}}$ is the indicator function equal 1 everywhere except at the origin $t = 0$ where it equals to 0. Concerning these states, the following simple proposition holds true.

PROPOSITION 5.1. *(i) At each instant $t > 0$, the discounted stock price is in one of the states (5.4) that satisfies the second order partial differential equation (3.8) with $r = 0$.*
(ii) With u given by (1.15) where $x_0 = 0$, we have

$$\grave{s}^\circ(x,t) = \int_{-\infty}^{\infty} u(x-y,\Delta t)\grave{s}^\circ(y,t+\Delta t)dy. \tag{5.5}$$

PROOF. (i) The required partial derivatives $\grave{s}_t^\circ$, $\grave{s}_x^\circ$ and $\grave{s}_{xx}^\circ$ are simply calculated. Thus (3.2) is easily verified.
(ii) By the obvious property $\grave{s}^\circ(x+\Delta x, t+\Delta t) = \grave{s}^\circ(\Delta x,\Delta t)\grave{s}^\circ(x,t)$ of the states (5.4), it suffices to show

$$\int_{-\infty}^{\infty} u(x,t)\grave{s}^\circ(-x,t)dx = 1.$$

But this is easily verified by (1.14) and (1.15). □

REMARK. By analogy to the prelimiting situation, we want to define the difference operator D in the state space. We depart from (4.4) and let $\Delta t_n = \frac{T}{N} \to 0$. The terms $\{\xi_{kn}^N\}_{k=1,\ldots,2^n}$ are negligible. According to (1.5) the denominator $2\sigma\sqrt{\Delta t_n}$ is the first approximation to the difference between two alternative states of the return. Then by the same arguments as above Brownian motion enters into consideration: we are led to define the limit on the left of (4.4) as $\frac{DS_t^\circ}{DW_t}$. This yields $\frac{DS^\circ}{DW} = S^\circ$, one of the first formulas of the Malliavin calculus, see [34], exercise 2.2.1 on p 107.

5.2. Self-financing strategies

Let us consider an investor who invests an amount $v \geq 0$ in the present market and then follows a trading strategy $\pi = (\Psi, \Phi)$ with portfolio components $\Psi = \{\Psi_t\}_{t\in[0,T]}$ and $\Phi = \{\Phi_t\}_{t\in[0,T]}$. The corresponding value process $V^\circ(\pi) = \Psi B^\circ + \Phi S^\circ$ is defined at $t \in [0,T]$ by

$$V^\circ(t;\pi) = \Psi(t)B^\circ(t) + \Phi(t)S^\circ(t) \tag{5.6}$$

with $v = V^\circ(0;\pi)$. If π is a self-financing strategy in the sense of the definition in section 4.1, then one can apply the integrating by parts formula of section 2.5 (with respect to the geometric Brownian motion instead of the ordinary Brownian motion in (2.9); this substitution is allowed by the chain rule (2.12)). This yields the same integral representation for discounted value process $\grave{V}^\circ(\pi) = \{\grave{V}^\circ(t;\pi)\}_{t\in[0,T]}$ as before: at each $t \in [0,T]$

$$\grave{V}^\circ(t;\pi) = v + \Phi \cdot \grave{S}_t^\circ, \tag{5.7}$$

cf proposition 4.1. Analogously to the trading in binary markets, the self-financing of a strategy $\pi = (\Psi, \Phi)$ means that the portfolio components $\Psi(t)$ and $\Phi(t)$ yield not only the market value of the holding at instant $t \in (0,T]$ (given by (5.6)) but also at an immediate future instant $t + \Delta t$:

$$V^\circ(t+\Delta t;\pi) = \Psi(t)B^\circ(t+\Delta t) + \Phi(t)S^\circ(t+\Delta t) \tag{5.8}$$

cf (4.1) and (4.2). Let us denote the possible states of the portfolio components $\Psi(t)$ and $\Phi(t)$ by $\{\psi(x,t), -\infty < x < \infty\}$ and $\{\phi(x,t), -\infty < x < \infty\}$ respectively, and the states of the discounted market value of this holding $\grave{V}^\circ(t;\pi)$ by $\{\grave{v}^\circ(x,t;\pi), -\infty < x < \infty\}$. Then (5.6) and (5.8) mean both

$$\grave{v}^\circ(x,t;\pi) = \psi(x,t) + \phi(x,t)\grave{s}^\circ(x,t) \tag{5.9}$$

and

$$\grave{v}^\circ(x+\Delta x, t+\Delta t;\pi) = \psi(x,t) + \phi(x,t)\grave{s}^\circ(x+\Delta x, t+\Delta t). \tag{5.10}$$

This fact has the following implication:

Proposition 5.2. *(i) At each instant $t > 0$, the discounted market value $\grave{V}^\circ(t;\pi)$ of a self-financing strategy π is in one of the states $\{\grave{v}^\circ(x,t;\pi), -\infty < x < \infty\}$ that satisfies the second order partial differential equation (3.8) with $r = 0$.*
(ii) With u given by (1.15) where $x_0 = 0$, we have

$$\grave{v}^\circ(x,t;\pi) = \int_{-\infty}^{\infty} u(x-y,\Delta t)\grave{v}^\circ(y,t+\Delta t;\pi)dy. \tag{5.11}$$

Proof. (i) may be obtained from (ii). Indeed, take (5.11) with $t + \Delta t = T$ and apply to both sides the operator $\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} - \mu\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. On the right the operator is allowed to act under the integral sign. But this yields 0, since $u(x-y,\tau)$ satisfies (3.8) with $r = 0$. As usual $\tau = T - t$ is time to maturity.
(ii) In (5.10) substitute $x + \Delta x$ by a variable y, multiply both sides by $u(x - y, \Delta t)$ and integrate with respect to dy. By (5.5) and (5.9) we get (5.11). □

Remark. Like in the previous remark at the end of section 5.1, we recall the prelimiting situation in which the stock component was expressed as $\Phi^N(t) = \frac{DV^N(t;\pi)}{DS^N(t)}$, see Clark's formula (4.6). This structure is retained as $N \to \infty$, thus giving the possibility to rewrite (5.7) in Clark's form

$$\grave{V}^\circ(t;\pi) = v + \frac{DV^\circ(\pi)}{DS^\circ} \cdot \grave{S}^\circ_t. \tag{5.12}$$

This is again an elementary formula in the Malliavin calculus, see [34], definition 1.2.1 on page 24. In the remaining part of this section more light will be shed on this formula, cf (5.15) below.

There is yet another relationship between the value process for a self-financing strategy and the stock price process. Namely, let $\grave{h}$ be a solution of the partial differential equation (3.11) with $r = \mu = 0$. Then for each $t \in [0,T]$ the following useful representation can be proved:

$$\grave{V}^\circ(t;\pi) = \grave{h}(\grave{S}^\circ(t),t). \tag{5.13}$$

In the next proposition the proof is provided in the form of the relationship between the states of the variables on the left and right-hand side.

Proposition 5.3. *(i) At each instant $t > 0$ and site x the states of the discounted stock price $\grave{S}^\circ(t)$ and the discounted value $\grave{V}^\circ(t;\pi)$ for a self-financing strategy π are related by*

$$\grave{v}^\circ(x,t;\pi) = \grave{h}(\grave{s}^\circ(x,t),t) \tag{5.14}$$

where $\grave{h}$ is as in (5.13).

Proof. By taking the relevant partial derivatives on the both sides of (5.14) we get

$$\frac{1}{2}\sigma^2\grave{v}^\circ_{xx} - \mu\grave{v}^\circ_x + \grave{v}^\circ_t = (\frac{1}{2}\sigma^2\grave{s}^\circ_{xx} - \mu\grave{s}^\circ_x + \grave{s}^\circ_t)\grave{h}_x + \frac{1}{2}\sigma^2\grave{s}^{\circ 2}_x\grave{h}_{xx} + \grave{h}_t$$

It remains now to recall the assertion (i) in proposition 5.1 and to apply (3.11) with $r = \mu = 0$ to the sum of the last two terms on the right. Indeed, by taking into consideration $\grave{s}^\circ_x = \grave{s}^\circ$ we get on the right-hand side $\frac{1}{2}\sigma^2 \grave{s}^\circ(x,t)^2 \grave{h}_{xx}(\grave{s}^\circ(x,t),t) + \grave{h}_t(\grave{s}^\circ(x,t),t)$ which equals 0 in view of the assumption that $\grave{h}$ satisfies (3.11) with $r = \mu = 0$. This means that (5.14) satisfies (3.8) with $r = 0$, cf proposition 5.2, assertion (i). □

Let us apply Itô's formula (2.13) to (5.13). Taking into consideration also the expression (2.11) for the quadratic variation and the chain rule (2.12), we get

$$\begin{aligned}\grave{V}^\circ(t;\pi) &= \grave{V}^\circ(0;\pi) + \int_0^t \grave{h}_x(\grave{S}^\circ(\theta),\theta)d\grave{S}^\circ(\theta) + \int_0^t \grave{h}_t(\grave{S}^\circ(\theta),\theta)d\theta \\ &+ \frac{1}{2}\sigma^2 \int_0^t \grave{h}_{xx}(\grave{S}^\circ(\theta),\theta)\grave{S}^\circ(\theta)^2 d\theta,\end{aligned}$$

cf [24], formula (1.12). But the last two terms vanish, since $\grave{h}$ satisfies (3.11) with $r = \mu = 0$. We thus have the integral representation (5.7) (or (5.12)) with

$$\Phi(t) = \frac{DV^\circ(t,\pi)}{DS^\circ(t)} = \grave{h}_x(\grave{S}^\circ(t),t). \tag{5.15}$$

Formula (5.13) is often presented in its nondiscounted form (see e.g. [24], formula (1.8), or [29], formula (5.8.36)):

$$V^\circ(t,\pi) = \bar{h}(S^\circ(t),t) \tag{5.16}$$

where $\bar{h}$ satisfies (3.11) with $r = -\mu$. Formula (5.15) for the stock component of the portfolio becomes $\Phi(t) = \bar{h}_x(\grave{S}^\circ(t),t)$. One can easily verify these claims by taking into consideration that $\bar{h}(x,t) = e^{rt}\grave{h}(e^{-rt}x,t)$. Of course, the factor e^{rt} means discounting, so (cf (5.1) and (5.3))

$$\grave{h}(\grave{S}^\circ(t),t) = \frac{\bar{h}(S^\circ(t),t)}{B^\circ(t)}. \tag{5.17}$$

5.3. Hedging strategies and option pricing

In the present section the same question as in section 4 arises whether the Black-Scholes market is complete or not. In order to formulate this question explicitly, consider again an investor whose goal is to attain at the terminal date T a certain wealth $W(T) = W(S^\circ(T))$ which is a certain function W of the stock price $W(S^\circ(T))$. According to (5.4) this wealth may be in one of the states

$$w^\circ(x,T) = W(s^\circ(x,T)), \quad -\infty < x < \infty. \tag{5.18}$$

If we now define the completeness of the present market similarly to the definition in section 4.4, then we get

Proposition 5.4. *The Black–Scholes market is complete: any desired wealth $W(T)$ of the above type is attainable with a certain initial endowment, since there is a uniquely defined self-financing strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$, called the hedging strategy against $W(T)$, whose value process $V^\circ(\pi^\circ) = \{V^\circ(t;\pi^\circ)\}_{t\in[0,T]}$ attains at the terminal date T the identity $V^\circ(T;\pi^\circ) = W(T)$. The necessary initial endowment is then $v = V^\circ(0;\pi^\circ)$.*

Proof. This follows from the explicit construction of the hedging strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ against $W(T)$ which is provided below.

The hedging strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ against $W(T)$ is constructed as follows. Let $\bar{h}$ be the solution of (3.11) with $r = -\mu$, subject to the boundary condition (3.14) where $H(x)$ is identified with $w^\circ(x,T)$ given by (5.18). According to proposition 3.4 this function has the representation (3.17) with $r = -\mu$ and with H substituted by W. Alternatively, we may work as in the previous section with $\grave{h}$ which solves (3.11) with $r = \mu = 0$ and is related to $\bar{h}$ via (5.17). Recall that $\grave{h}_x = \bar{h}_x$. Use the notations

$$\psi^\circ(x,t) = \grave{h}(x,t) - x\grave{h}_x(x,t) = e^{-rt}\bar{h}(e^{rt}x,t) - x\bar{h}_x(x,t)$$

and

$$\phi^\circ(x,t) = \grave{h}_x(x,t) = \bar{h}_x(x,t)$$

to define the strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ with $\Psi^\circ(t) = \psi^\circ(\grave{S}^\circ(t),t)$ and $\Phi^\circ(t) = \phi^\circ(\grave{S}^\circ(t),t)$. By definition (5.6) the market value of the latter holding is simply determined:

$$V^\circ(t,\pi^\circ) = \psi^\circ(\grave{S}^\circ(t),t)B^\circ(t) + \phi^\circ(\grave{S}^\circ(t),t)S^\circ(t) = \bar{h}(S^\circ(t),t),$$

cf (5.16). This is indeed the hedging strategy, since at maturity the market value $V^\circ(T,\pi^\circ) = \bar{h}(S^\circ(T),T)$ amounts to the desired wealth $W(S^\circ(T))$ in virtue of the boundary condition fixed above. Thus the desired wealth is attained. In view of (3.17) with $r = -\mu$, the necessary initial endowment amounts to

$$v = V^\circ(0;\pi^\circ) = e^{-rT}\int_{-\infty}^{\infty} g(y,0,\sigma^2 T)H(xe^{y+rT-\frac{1}{2}\sigma^2 T})dy. \tag{5.19}$$

with H substituted by W. □

Observe that in the special case of $H(x) = (x-K)^+$ the right-hand side reduces to the integral in (4.27). This is not just a coincidence, as the reader might guess knowing the methodology of option pricing by means of a hedging strategy that duplicates the payoff (see part I, section 5, or the present part, section 4.4). We conclude this section with some more details on this.

We are now going to use formula (5.19) for solving the problem of pricing contingent claims as is already described in section 4.4. Let the payoff function of a contingent claim be determined by a certain nonnegative function H of the stock price at maturity $S^\circ(T)$, i.e. $H(T) = H(S^\circ(T))$. According to the procedure developed in section 4.4, we have

PROPOSITION 5.5. *In the Black–Scholes market the fair price $C(H)$ of a contingent claim with the payoff function $H(T) = H(S^\circ(T))$ is identified with the right-hand side of (5.19).*

As was already mentioned, formula (5.19) applied to the special payoff $(x-K)^+$ determines the fair price C of the European call option, which coincides with the integral in (4.27). It is now easy to present C in the form (5.20) below, suitable for calculations by using tables of the standard normal distribution. This is called the *Black–Scholes option valuation formula.*

COROLLARY 5.6. *In the Black–Scholes market the fair price C of the European call option is presented as follows*

$$C = sG\left(\frac{\log\frac{s}{K} + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - e^{-rT}KG\left(\frac{\log\frac{s}{K} + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \quad (5.20)$$

with G the standard normal distribution, cf (1.12).

PROOF. The right-hand side in (4.27), equal to

$$\int_{-\infty}^{\infty} g(y; \frac{1}{2}\sigma^2 T, \sigma^2 T)\left(se^{-y} - \grave{K}\right)^+ dy,$$

reduces to

$$\begin{aligned} & \int_{-\infty}^{\log\frac{s}{K}} g(y; -rT + \frac{1}{2}\sigma^2 T, \sigma^2 T)\left(se^{-(y+rT)} - \grave{K}\right) dy \\ = & \int_{-\infty}^{\log\frac{s}{K}} sg(y; -rT - \frac{1}{2}\sigma^2 T, \sigma^2 T) dy \\ & -\grave{K}\int_{-\infty}^{\log\frac{s}{K}} g(y; -rT + \frac{1}{2}\sigma^2 T, \sigma^2 T) dy. \end{aligned}$$

By (1.12) the right-hand side coincides with that of (5.20). The proof is complete. □

Using similar considerations one can specify the hedging strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ against the European call option by the portfolio components at $t \in [0, T]$ and $\tau = T - t$

$$\Psi^\circ(t) = -e^{-rt}KG\left(\frac{\log\frac{S^\circ(t)}{K} + r\tau - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right)$$

and

$$\Phi^\circ(t) = G\left(\frac{\log\frac{S^\circ(t)}{K} + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right).$$

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